

Number Fields: Example Sheet 1 of 3

1. Find the minimal polynomials over \mathbb{Q} of

$$(1+i)\sqrt{3}, \quad i+\sqrt{3}, \quad 2\cos(2\pi/7).$$

2. Which of the following are algebraic integers?

$$\sqrt{5}/\sqrt{2}, \quad (1+\sqrt{3})/2, \quad (\sqrt{3}+\sqrt{7})/2, \quad \frac{3+2\sqrt{6}}{1-\sqrt{6}}, \quad (1+\sqrt[3]{10}+\sqrt[3]{100})/3, \quad 2\cos(2\pi/19).$$

3. Let $d > 1$ be an integer. Show that the only units in the ring

$$\mathbb{Z}[\sqrt{-d}] = \{a + b\sqrt{-d} : a, b \in \mathbb{Z}\}$$

are ± 1 .

4. Let K be a number field. Show that every extension L/K of degree 2 is of the form $L = K(\sqrt{a})$ with $a \in K^*$, $a \notin (K^\times)^2$. Show further that there is an isomorphism $K(\sqrt{a}) \cong K(\sqrt{b})$ inducing the identity on K if and only if $a/b \in (K^\times)^2$.

5. Let $K = \mathbb{Q}(\theta)$ where θ is a root of $X^3 - 2X + 6$. Show that $[K : \mathbb{Q}] = 3$ and compute $N_{K/\mathbb{Q}}(\alpha)$ and $\text{tr}_{K/\mathbb{Q}}(\alpha)$ for $\alpha = n - \theta$, $n \in \mathbb{Z}$ and $\alpha = 1 - \theta^2, 1 - \theta^3$.

6. Let $K = \mathbb{Q}(\delta)$ where $\delta = \sqrt[3]{d}$ and $d \neq 0, \pm 1$ is a square-free integer. Show that $\text{disc}(1, \delta, \delta^2) = -27d^2$. By calculating the traces of $\theta, \delta\theta, \delta^2\theta$, and the norm of θ , where $\theta = u + v\delta + w\delta^2$ with $u, v, w \in \mathbb{Q}$, show that the ring of integers \mathcal{O}_K of K satisfies

$$\mathbb{Z}[\delta] \subset \mathcal{O}_K \subset \frac{1}{3}\mathbb{Z}[\delta].$$

7. Let $f(X) \in \mathbb{Q}[X]$ be a monic irreducible polynomial of degree n , and $\theta \in \mathbb{C}$ a root of f .

(i) Show that $\text{disc}(f) = (-1)^{\binom{n}{2}} N_{K/\mathbb{Q}}(f'(\theta))$ where $K = \mathbb{Q}(\theta)$.

(ii) Let $f(X) = X^n + aX + b$. Write down the matrix representing multiplication by $f'(\theta)$ with respect to the basis $1, \theta, \dots, \theta^{n-1}$ for K . Hence show that

$$\text{disc}(f) = (-1)^{\binom{n}{2}} ((1-n)^{n-1} a^n + n^n b^{n-1}).$$

8. Compute an integral basis for \mathcal{O}_K in the cases $K = \mathbb{Q}[X]/(X^3 + X + 1)$ and $K = \mathbb{Q}[X]/(X^3 - X - 4)$.

9. Suppose that $K = \mathbb{Q}(\theta)$ is a number field of degree $n = r + 2s$ in the usual notation (r is the number of real embeddings of K and s the number of pairs of complex conjugate embeddings). Show that the sign of the discriminant D_K is $(-1)^s$.

10. Let $K = \mathbb{Q}(i, \sqrt{2})$. By computing the relative traces $\text{tr}_{K/k}(\theta)$ where k runs through the three quadratic subfields of K , show that the algebraic integers θ in K have the form $\frac{1}{2}(\alpha + \beta\sqrt{2})$, where $\alpha = a + ib$ and $\beta = c + id$ are Gaussian integers. By considering $N_{K/k}(\theta)$ where $k = \mathbb{Q}(i)$ show that

$$\begin{aligned} a^2 - b^2 - 2c^2 + 2d^2 &\equiv 0 \pmod{4}, \\ ab - 2cd &\equiv 0 \pmod{2}. \end{aligned}$$

Hence prove that an integral basis for \mathcal{O}_K is $1, i, \sqrt{2}, \frac{1}{2}(1+i)\sqrt{2}$, and calculate the discriminant D_K .

11. Let K be a quadratic field and $I \subset \mathcal{O}_K$ an ideal. Show that $I = (\alpha, \beta)$ for some $\alpha \in \mathbb{Z}$ and $\beta \in \mathcal{O}_K$. Let $c = \gcd(\alpha^2, \alpha \text{tr}_{K/\mathbb{Q}}\beta, N_{K/\mathbb{Q}}\beta)$. By computing the norm and trace show that $\frac{\alpha\beta}{c} \in \mathcal{O}_K$. Deduce that $(\alpha, \beta)(\alpha, \beta')$ is principal, where $\beta\beta' = N_{K/\mathbb{Q}}\beta$.