NUMBER FIELDS, EXX. SHEET 1

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(1*) Which of the following are algebraic integers: $\sqrt{5}$, $\sqrt{5}/\sqrt{2}$, $(1+\sqrt{5})/2$, $\sqrt{(1+\sqrt{5})/\sqrt{2}}$, $(1+\sqrt{3})/2$, $(1+\sqrt{-3})/2$?

(2*) Define *number field*, *algebraic number*, *algebraic integer*. Prove that the algebraic integers in a given number field K form a ring $\mathcal{O}_K$. [Bookwork, but do it anyway.]

(3*) If K is a number field and $a \in K$, what are the conjugates of a? What are the conjugates of $2^{1/3}$? Explain how to use the conjugates of a to check whether a is an algebraic integer.

(4*) Define the trace bilinear pairing $T = T_{K/k}$ of a finite field extension K/k, and use it to prove that $\mathcal{O}_K$ is a f.g. $\mathbb{Z}$-module when K is a number field. [Bookwork, but do it anyway.]

(5*) Define the *discriminant* of a subring R of $\mathcal{O}_K$ such that K is the fraction field of R. Show that if $R = \mathbb{Z}[a]$ and f is the minimal polynomial of a, then the discriminant of R equals that of f.

Assuming that D is a square-free integer and is prime to 3, compute a $\mathbb{Z}$-basis of $\mathcal{O}_K$ when $\theta = D^{1/3}$ and $K = \mathbb{Q}(\theta)$.

[In an exam you would be told to take for granted the useful result that if $x = a + b\theta + c\theta^2$ with $a, b, c \in \mathbb{Q}$, then the elementary symmetric polynomials e_1, e_2, e_3 in x and its conjugates are $e_1 = 3a$, $e_2 = 3a^2 - 3Dbc$, $e_3 = a^3 + Db^3 + D^2c^3 - 3Dabc$.]

(6*) Suppose that d is a square-free integer, $d \neq 0, 1$. Describe the ring of integers in the quadratic field $K = \mathbb{Q}(\sqrt{d})$ and compute the discriminant D_K of this field. Show that if $f \in \mathbb{Z}[x]$ is a monic quadratic polynomial of discriminant D_K, then $\mathcal{O}_K \equiv \mathbb{Z}[x]/(f)$.

(7) Suppose that K is a number field of degree $n = r + 2s$ in the usual notation (r is the number of real embeddings of K and s the number of pairs of complex embeddings). Show that the sign of the discriminant D_K is $(-1)^s$.

(8) Prove Stickelberger's criterion, that $D_K \equiv 0, 1 \pmod 4$. [1]

[1] [Hint: Suppose first that $K/\mathbb{Q}$ is Galois, with group G. If $(x_1, ..., x_n)$ is a $\mathbb{Z}$-basis of $\mathcal{O}_K$ and $\sigma_1, ..., \sigma_n$ are the real and complex embeddings of K, then $D_K = \Delta^2$, where $\Delta = \det((\sigma_i(x_j))$. Write $\Delta = P - N$, where P corresponds to the even permutations of n things and N to the odd ones. So $D_K = (P+N)^2 - 4PN$. Since $K/\mathbb{Q}$ is Galois, the conjugates $\sigma_i(x)$, for any $x \in K$, are exactly the images $g_i(x)$ of x under the various elements g_i of G. Deduce that $P + N$ and PN are G-invariant, so in $\mathbb{Q}$. Notice that P, N are algebraic integers, so $P + N, PN \in \mathbb{Z}$.

For the general case, embed K in a Galois closure $L/\mathbb{Q}$.]

(9) Show that $f = x^3 - x + 2$ is irreducible over $\mathbb{Q}$ and that the ring of integers in $\mathbb{Q}[x]/(f)$ is $\mathbb{Z}[x]/(f)$.

(10) Suppose that p is an odd prime and that $\zeta = \zeta_p = \exp(2\pi i/p)$. Put $K = \mathbb{Q}(\zeta)$ and $A = \mathbb{Z}[\zeta]$, a subring of $\mathcal{O}_K$ (why?).

(i) Write down the minimal polynomial of ζ and the conjugates of ζ.

(ii) Show that if r, s are prime to p, then $(\zeta^r - 1)/(\zeta^s - 1)$ is a unit in A.

(iii) Show that $\zeta - 1$ is a prime element of A and that $(p) = (\zeta - 1)^{p-1}$ as principal ideals in A.

(iv) Deduce that $P = (\zeta^s - 1)$ is the only prime ideal of A that lies over p, and that the local ring A_P is a DVR.

(v) Show that the discriminant of A is (up to sign) a power of p.

(vi) Deduce that A_Q is a DVR for all prime ideals Q of A, and that therefore $A = \mathcal{O}_K$.

(11*) Find factorizations of ideals (in rings of integers $\mathcal{O}_K$) of the form $(p) = P_1^{e_1}...P_r^{e_r}$ in the following cases:

$K = \mathbb{Q}(\sqrt{17})$, $p = 2, 3, 5$; $K = \mathbb{Q}(\zeta_5)$, $p = 2, 3, 5$. [That is, find the integers r and the exponents e_i, and the norm of each P_i.]

References

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