1. Which of the following propositions are tautologies?

(i) $(p_1 \Rightarrow (p_2 \Rightarrow p_3)) \Rightarrow (p_2 \Rightarrow (p_1 \Rightarrow p_3))$

(ii) $((p_1 \vee p_2) \wedge (p_1 \vee p_3)) \Rightarrow (p_2 \vee p_3)$

(iii) $(p_1 \Rightarrow (\neg p_2)) \Rightarrow (p_2 \Rightarrow (\neg p_1))$

2. Write down a proof of $\perp \Rightarrow q$. Use this to write down a proof of $p \Rightarrow q$ from $\neg p$.

3. Use the Deduction Theorem to show that $p \vdash \neg\neg p$.

4. Show that $\{p, q\} \vdash (p \wedge q)$ in three different ways: by writing down a proof, by using the Deduction Theorem and by using the Completeness Theorem.

5. Give propositions p and q for which $(p \Rightarrow q) \Rightarrow \neg(q \Rightarrow p)$ is a tautology.

6. Three people each have a set of beliefs: a consistent deductively closed set. Show that the set of propositions that they all believe is also consistent and deductively closed. Must the set of propositions that a majority believe be consistent? Must it be deductively closed?

7. Show that the third axiom cannot be deduced from the first two. In other words, show that (for some p) there is no proof of $(\neg\neg p \Rightarrow p)$ that uses only the first two axioms and modus ponens.

8. Let $t_1, t_2, \ldots$ be propositions such that, for every valuation v, there exists $n \in \mathbb{N}$ with $v(t_n) = 1$. Show that there exists $n \in \mathbb{N}$ such that $\vdash (t_1 \vee t_2 \vee \ldots \vee t_n)$.

9. Let a and c be propositions such that $a \vdash c$. Show that there is a proposition b, in which the only primitive propositions appearing are those that appear in both a and c, such that $a \vdash b$ and $b \vdash c$.

10. Two sets S, T of propositions are equivalent if $S \vdash t$ for every $t \in T$ and $T \vdash s$ for every $s \in S$. A set S of propositions is independent if for every $s \in S$ we have $S \setminus \{s\} \nvdash s$. Show that every finite set of propositions has an independent subset equivalent to it. Give an infinite set of propositions that has no independent subset equivalent to it. Show, however, that for every countable set of propositions there exists an independent set equivalent to it.

11. Give an explicit function $f\colon \mathbb{N} \to \mathbb{N}$ such that every tautology of length n has a proof that is at most $f(n)$ lines long.

12. A set S of propositions is a chain if for any distinct $p, q \in S$ we have $p \vdash q$ or $q \vdash p$ but not both. Write down an infinite chain. If the set of primitive propositions is allowed to be uncountable, can there exist an uncountable chain?

13. Let p be a tautology not involving the symbol $\perp$. Must it be possible to prove p without using the third axiom?