1. Show that \( \{\text{Inf}, \text{Sep}\} \vdash \text{Emp} \). Does \( \text{Inf} \vdash \text{Emp} \)? Does \( \text{Sep} \vdash \text{Emp} \)? [N.B. You should interpret \( \{\text{Inf}, \text{Sep}\} \) as “the collection of formulae in the language of sets consisting of the Axiom of Infinity and every instance of the Axiom of Separation” etc.]

2. Show that \( \text{Rep} \vdash \text{Sep} \). Show also that \( \{\text{Emp}, \text{Pow}, \text{Rep}\} \vdash \text{Pair} \).

3. Write down sentences in the language of sets to express the assertions that, for any two sets \( x \) and \( y \), the product \( x \times y \) and the set of all functions from \( x \) to \( y \) exist. Indicate how to deduce these sentences from the axioms of ZF.

4. Write down a formula \( p \) in the language of sets with \( \text{FV}(p) = \{x\} \) that says “\( x \) is a (von Neumann) ordinal”. What should the von Neumann ordinal \( \omega^2 \) be? Why do the axioms of ZF prove that it is a set?

5. Is it true that if \( x \) is a transitive set then the relation \( \in \) on \( x \) is a transitive relation? Does the converse hold?

6. Let \( \varepsilon\text{-Ind} \) denote the principle of \( \varepsilon \)-induction and let \( p \) be the formula \( (\forall y)((x \in y) \Rightarrow (\exists z)((z \in y) \land (z \cap y = \emptyset))) \). Show that \( (\text{ZF}\setminus\{\text{Fdn}\}) \cup \{\varepsilon\text{-Ind} \} \vdash (\forall x)p \) and hence that \( (\text{ZF}\setminus\{\text{Fdn}\}) \cup \{\varepsilon\text{-Ind} \} \vdash \text{Fdn} \).

7. What is the rank of \( \{2, 3, 6\} \)? What is the rank of \( \{\{2, 3\}, \{6\}\} \)? Work out the ranks of \( \mathbb{Z}, \mathbb{Q} \) and \( \mathbb{R} \), using your favourite constructions of these objects from \( \omega \).

8. A set \( x \) is called hereditarily finite if each member of \( \text{TC}(\{x\}) \) is finite. Prove that the class \( \text{HF} \) of hereditarily finite sets coincides with \( V_{\omega} \). Which of the axioms of ZF are satisfied in the structure \( \text{HF} \)?

9. Which of the axioms of ZF are satisfied in the structure \( V_{\omega+\omega} \)?

10. If ZF is consistent then, by Downward Löwenheim-Skolem, it has a countable model. Doesn’t this contradict the fact that, for example, the power-set of \( \omega \) is uncountable?

11. Assume that ZF is consistent. We extend the language of ZF by adding new constants \( \alpha_1, \alpha_2, \ldots \), and extend the axioms of ZF by adding (for each \( n \)) the assertions that \( \alpha_n \) is an ordinal and that \( \alpha_{n+1} < \alpha_n \). Explain why this theory has a model. In this model of ZF, haven’t we contradicted the fact that the ordinals are “well-ordered” (that is to say, that each non-empty set of ordinals has a least element)?

12. Prove (in ZF) that a countable union of countable sets cannot have cardinality \( \aleph_2 \).

13. Is every countable model of PA isomorphic to \( \mathbb{N} \)? (What does “isomorphic” mean?)

14. Is \( \text{PA} \cup \{\neg\text{Con}(\text{PA})\} \) consistent? Is it \( \omega \)-consistent?

15. Show that the function \( f(n) = 2^n \) is definable in the language of PA; that is to say, show that there is a formula \( p \) in the language of PA with \( \text{FV}(p) = \{x, y\} \) such that, for all \( m, n \in \mathbb{N} \), \( (m, n) \in p_n \) iff \( n = 2^m \).