Set Theory and Logic, Michaelmas 2016, Sheet 2: Posets

October 27, 2016

'+' signifies a question you shouldn't have trouble with; '☠' means what you think it means.

(i)

For $n \in \mathbb{N}$, how many antisymmetrical binary relations are there on a set of cardinality n? How many binary relations satisfying *trichotomy*: $(\forall xy)(R(x,y) \vee R(y,x) \vee x = y)$? How are your two answers related?

[That was the version that was circulated on the printed sheet. A better version asks for the numbers of *symmetric* relations and the number of *antisymmetric trichotomous* relations.]

(ii)

Consider the set of equivalence relations on a fixed set partially ordered by $\subseteq$. Show that it is a lattice. Must it be distributive? Is it complete?

(iii)

Cardinals: Recall that $\alpha \cdot \beta$ is $|A \times B|$ where $|A| = \alpha$ and $|B| = \beta$. Show that a union of α disjoint sets each of size β has size $\alpha \cdot \beta$. Explain your use of AC.

(iv)

Let $\langle A, \leq \rangle$ and $\langle B, \leq \rangle$ be total orderings with $\langle A, \leq \rangle$ isomorphic to an initial segment of $\langle B, \leq \rangle$ and $\langle B, \leq \rangle$ isomorphic to a terminal segment of $\langle A, \leq \rangle$. Show that $\langle A, \leq \rangle$ and $\langle B, \leq \rangle$ are isomorphic.

(v)

(Mathematics Tripos Part II 2002:B2:11b, modified).

1. Let U be an arbitrary set and $\mathcal{P}(U)$ be the power set of U. For X a subset of $\mathcal{P}(U)$, the **dual** $X^\vee$ of X is the set $\{y \subseteq U : (\forall x \in X)(y \cap x \neq \emptyset)\}$.
2. Is the function $X \mapsto X^\vee$ monotone? Comment.
3. By considering the poset of those subsets of $\mathcal{P}(X)$ that are subsets of their duals, or otherwise, show that there are sets $X \subseteq U$ with $X = X^\vee$.

4. $X^{\vee\vee}$ is clearly a superset of X, in that it contains every superset of every member of X. What about the reverse inclusion? That is, do we have $Y \in X^{\vee\vee} \to (\exists Z \in X)(Z \subseteq Y)$?
5. Is $A^{\vee\vee\vee}$ always equal to $A^{\vee}$?

(vi)

Use Zorn's Lemma to prove that

(i) every partial ordering on a set X can be extended to a total ordering of X;

(ii) for any two sets A and B, there exists either an injection $A \hookrightarrow B$ or an injection $B \hookrightarrow A$.

(vii)

(Tripos IIA 1998 p 10 q 7)

Let $\langle P, \leq_P \rangle$ be a chain-complete poset with a least element, and $f : P \to P$ an order-preserving map. Show that the set of fixed points of f has a least element and is chain-complete in the ordering it inherits from P. Deduce that if $f_1, f_2, \ldots, f_n$ are order-preserving maps $P \to P$ which commute with each other (i.e. $f_i \circ f_j = f_j \circ f_i$ for all i, j), then they have a common fixed point. Show by an example that two order-preserving maps $P \to P$ which do not commute with each other need not have a common fixed point.

(viii)

$\mathbb{N} \rightharpoonup \mathbb{N}$ is the set of partial functions from $\mathbb{N}$ to $\mathbb{N}$, thought of as sets of ordered pairs and partially ordered by $\subseteq$.

Is it complete? Directed-complete? Separative? Which fixed point theorems are applicable?

For each of the following functions $\Phi : (\mathbb{N} \rightharpoonup \mathbb{N}) \to (\mathbb{N} \rightharpoonup \mathbb{N})$, determine (a) whether Φ is order-preserving, and (b) whether it has a fixed point:

(i) $\Phi(f)(n) = f(n) + 1$ if $f(n)$ is defined, undefined otherwise.

(ii) $\Phi(f)(n) = f(n) + 1$ if $f(n)$ is defined, $\Phi(f)(n) = 0$ otherwise.

(iii) $\Phi(f)(n) = f(n-1) + 1$ if $f(n-1)$ is defined, $\Phi(f)(n) = 0$ otherwise.

(ix)

Players `I` and `II` alternately pick elements (`I` plays first) from a set A (repetitions allowed: A does not get used up) thereby jointly constructing an element s of A^ω, the set of ω-sequences from A. Every subset $X \subseteq A^\omega$ defines a game $G(X)$ which is won by player `I` if $s \in X$ and by `II` otherwise. Give A the discrete topology and A^ω the product topology.

By considering the poset of partial functions $A^{<\omega} \to \{\mathtt{I}, \mathtt{II}\}$ ($A^{<\omega}$ is the set of finite sequences from A) or otherwise prove that if X is open then one of the two players must have a winning strategy.

(x)

$\mathbb{R} = \langle 0, 1, +\times, \leq \rangle$ is a field. Consider the product $\mathbb{R}^{\mathbb{N}}$ of countably many copies thereof, with operations defined pointwise. Let $\mathcal{U}$ be an ultrafilter on $\mathbb{N}$ and consider $\mathbb{R}^{\mathbb{N}}/\mathcal{U}$. Prove that it is a field. Is it archimedean?

(xi)

(i)$^+$ How many order-preserving maps $\mathbb{R} \to \mathbb{R}$ are there?

(ii)☠ Let $\langle X, \leq_X \rangle$ be a total order with no nontrivial order-preserving injection $X \to X$. Must X be finite?