Questions marked with a ‘*’ may be skipped by the nervous.

(i)
Write down subsets of \( \mathbb{R} \) of order types \( \omega + \omega \), \( \omega^2 \) and \( \omega^3 \) in the inherited order.

(ii)
Which of the following are true?
(a) \( \alpha + \beta \) is a limit ordinal iff \( \beta \) is a limit ordinal;
(b) \( \alpha \cdot \beta \) is a limit ordinal iff \( \alpha \) or \( \beta \) is a limit ordinal;
(c) Every limit ordinal is of the form \( \alpha \cdot \omega \);
(d) Every limit ordinal is of the form \( \omega \cdot \alpha \).

For these purposes 0 is a limit ordinal.

(iii)
Consider the two functions \( \text{On} \rightarrow \text{On} \): \( \alpha \mapsto 2^\alpha \) and \( \alpha \mapsto \alpha^2 \). Are they normal?

(iv)
Prove the converse to lemma ??: if \( \langle X, <_X \rangle \) is a total order satisfying “every subordering is isomorphic to an initial segment” then it is a wellordering.

(v)
What is the smallest ordinal you can not embed in the reals in the style of question (i)?

(vi)
Prove that every [nonzero] countable limit ordinal has cofinality \( \omega \). What about \( \omega_1 \)?
Recall the recursive definition of ordinal exponentiation:

\[ \alpha^0 = 1; \quad \alpha^{\beta + 1} = \alpha^\beta \cdot \alpha, \quad \text{and} \quad \alpha^{\sup(B)} = \sup(\{\alpha^\beta : \beta \in B\}). \]

Ordinal addition corresponds to disjoint union [of wellorderings], ordinal multiplication corresponds to lexicographic product, and ordinal exponentiation corresponds to . . . ?

Give a definition of a suitable operation on wellorderings and show that your definition conforms to the spec: \( \alpha^{\beta+\gamma} = \alpha^\beta \cdot \alpha^\gamma \).

Let \( \{X_i : i \in I\} \) be a family of sets, and \( Y \) a set. For each \( i \in I \) there is an injection \( X_i \hookrightarrow Y \). Give an example to show that there need not be an injection \( (\bigcup_{i \in I} X_i) \hookrightarrow Y \). But what if the \( X_i \) are nested? [That is, \( (\forall i, j \in I)(X_i \subseteq X_j \lor X_j \subseteq X_i) \).]

Prove that every ordinal of the form \( \omega^\alpha \) is indecomposable: \( \gamma + \beta = \omega^\alpha \rightarrow \gamma = \omega^\alpha \lor \beta = \omega^\alpha \).

Show that an arbitrary intersection of transitive relations is transitive. The transitive closure \( R^* \) (sometimes written ‘\( tr(R) \)’) is the \( \subseteq \)-least transitive relation \( \supseteq R \).

Let \( (X, R) \) be a wellfounded binary structure, with rank function \( \rho \). Prove that \( (\forall x \in X)(\forall \alpha < \rho(x))(\exists y)(R^*(y, x) \land \rho(y) = \alpha) \).

Let \( \{X_i : i \in \mathbb{N}\} \) be a nested family of sets of ordinals.

(a) Give an example to show that the order type of \( \bigcup_{i \in \mathbb{N}} X_i \) need not be the sup of the order types of the \( X_i \).

(b) What condition do you need to put on the inclusion relation between the \( X_i \) to ensure that the order type of \( \bigcup_{i \in \mathbb{N}} X_i \) is the sup of the order types of the \( X_i \)?

(c) Show that the ordered set of the rationals can be obtained as the union of a suitably chosen \( \omega \)-chain of some of its finite subsets.

Using the uniqueness of subtraction for ordinals, and the division algorithm for normal functions, show that every ordinal can be expressed uniquely as a sum

\[ \omega^{\alpha_1} \cdot a_1 + \omega^{\alpha_2} \cdot a_2 + \cdots + \omega^{\alpha_n} \cdot a_n \]

where all the \( a_i \) are finite, and where the \( \alpha_1 \) and the \( a_i \) are strictly decreasing.
(xiii)

Let $f$ be a function from countable [nonzero] limit ordinals to countable ordinals satisfying $f(\alpha) < \alpha$ for all (countable limit) $\alpha$. ($f$ is “pressing-down”.) Can $f$ be injective?