Set Theory and Logic, Michaelmas 2016, Sheet 1: Ordinals and Induction

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Questions marked with a '*' may be skipped by the nervous.

(i)

Write down subsets of $\mathbb{R}$ of order types $\omega + \omega$, ω^2 and ω^3 in the inherited order.

(ii)

Which of the following are true?

(a) $\alpha + \beta$ is a limit ordinal iff β is a limit ordinal;
(b) $\alpha \cdot \beta$ is a limit ordinal iff α or β is a limit ordinal;
(c) Every limit ordinal is of the form $\alpha \cdot \omega$;
(d) Every limit ordinal is of the form $\omega \cdot \alpha$.

For these purposes 0 is a limit ordinal.

(iii)

Consider the two functions $On \to On$: $\alpha \mapsto 2^\alpha$ and $\alpha \mapsto \alpha^2$. Are they normal?

(iv)

Prove the converse to lemma **??**: if $\langle X, <_X \rangle$ is a total order satisfying "every subordering is isomorphic to an initial segment" then it is a wellordering.

(v)

What is the smallest ordinal you can not embed in the reals in the style of question (i)?

(vi)

Prove that every [nonzero] countable limit ordinal has cofinality ω. What about ω_1?

(vii)*

Recall the recursive definition of ordinal exponentiation:

$$\alpha^0 = 1;\ \alpha^{\beta+1} = \alpha^\beta \cdot \alpha, \text{ and } \alpha^{sup(B)} = \sup(\{\alpha^\beta : \beta \in B\}).$$

Ordinal addition corresponds to disjoint union [of wellorderings], ordinal multiplication correponds to lexicographic product, and ordinal exponentiation corresponds to ...? Give a definition of a suitable operation on wellorderings and show that your definition conforms to the spec: $\alpha^{\beta+\gamma} = \alpha^\beta \cdot \alpha^\gamma$.

(viii)

Let $\{X_i : i \in I\}$ be a family of sets, and Y a set. For each $i \in I$ there is an injection $X_i \hookrightarrow Y$. Give an example to show that there need not be an injection $(\bigcup_{i \in I} X_i) \hookrightarrow Y$. But what if the X_i are nested? [That is, $(\forall i, j \in I)(X_i \subseteq X_j \vee X_j \subseteq X_i)$.]

(ix)

Prove that every ordinal of the form ω^α is **indecomposible**: $\gamma + \beta = \omega^\alpha \ \rightarrow \ \gamma = \omega^\alpha \ \vee \ \beta = \omega^\alpha$.

(x)

Show that an arbitrary intersection of transitive relations is transitive. The **transitive closure** R^* (sometimes written '$tr(R)$') is the $\subseteq$-least transitive relation $\supseteq R$.

Let $\langle X, R \rangle$ be a wellfounded binary structure, with rank function ρ. Prove that $(\forall x \in X)(\forall \alpha < \rho(x))(\exists y)(R^*(y, x) \wedge \rho(y) = \alpha)$.

(xi)

Let $\{X_i : i \in \mathbb{N}\}$ be a nested family of sets of ordinals.

(a) Give an example to show that the order type of $\bigcup_{i \in \mathbb{N}} X_i$ need not be the sup of the order types of the X_i.
(b) What condition do you need to put on the inclusion relation between the X_i to ensure that the order type of $\bigcup_{i \in \mathbb{N}} X_i$ is the sup of the order types of the X_i?
(c) Show that the ordered set of the rationals can be obtained as the union of a suitably chosen ω-chain of some of its finite subsets.

(xii)

Using the uniqueness of subtraction for ordinals, and the division algorithm for normal functions, show that every ordinal can be expressed uniquely as a sum

$$\omega^{\alpha_1} \cdot a_1 + \omega^{\alpha_2} \cdot a_2 + \cdots \omega^{\alpha_n} \cdot a_n$$

where all the a_i are finite, and where the α_1 and the a_i are strictly decreasing.

(xiii)

Let f be a function from countable [nonzero] limit ordinals to countable ordinals satisfying $f(\alpha) < \alpha$ for all (countable limit) α. (f is "*pressing-down*".) Can f be injective?