Logic and Set Theory Examples 2

PTJ Lent 2016

Important Note: There are probably more questions on this example sheet than any one student will wish to attempt, and they are not intended to be all of the same level of difficulty. Some (marked by − signs) are mere five-finger exercises to ensure that you have understood the basic definitions, and may safely be omitted if you are **confident** that you have understood them. Others (marked by + signs) are more challenging and/or slightly off the syllabus, and are intended for students who wish to explore the subject a bit more widely than can be done in the course.

Comments on and/or corrections to the questions on this sheet are always welcome, and may be e-mailed to me at `ptj@dpmms.cam.ac.uk`.

− **1**. Which of the following propositional formulae are tautologies?

$$\text{(i)}\quad ((p\Rightarrow(q\Rightarrow r))\Rightarrow(q\Rightarrow(p\Rightarrow r))); \qquad \text{(ii)}\quad (((p\Rightarrow q)\Rightarrow r)\Rightarrow((q\Rightarrow p)\Rightarrow r));$$
$$\text{(iii)}\quad (((p\Rightarrow q)\Rightarrow p)\Rightarrow p); \qquad \text{(iv)}\quad ((p\Rightarrow(p\Rightarrow q))\Rightarrow p)\ .$$

− **2**. Use the Deduction Theorem to show that the converse of the third axiom (i.e. the formula $(p\Rightarrow\neg\neg p)$) is a theorem of the propositional calculus.

3. Let t be a propositional formula not involving the constant $\bot$, and let $t' = t[\bot/p]$ be the formula obtained from t by substituting $\bot$ for all occurrences of a particular propositional variable p in t. Suppose that t' is a tautology but t is not; show that any proof of t' in the propositional calculus must involve an instance of the third axiom. Does this result remain true (a) if t is allowed to contain occurrences of $\bot$, or (b) if $\bot$ is replaced by $\top$?

4. Write down a proof of $(\bot\Rightarrow q)$ in the propositional calculus [hint: observe the result of question 3], and hence write down a deduction of $(p\Rightarrow q)$ from $\{\neg p\}$.

5. Show that if there is a deduction of t from $S\cup\{s\}$ in n lines (that is, consisting of n consecutive formulae), then $(s\Rightarrow t)$ is deducible from S in at most $3n+2$ lines. Show further that there is a deduction of $\bot$ from $\{((p\Rightarrow q)\Rightarrow p), (p\Rightarrow\bot)\}$ in 16 lines [hint: use question 4], and hence calculate an upper bound for the length of a proof of the tautology of question 1(iii).

6. The beliefs of each member of a finite non-empty set I of individuals are represented by a consistent, deductively closed set of propositional formulae (in some fixed language $\mathcal{L}(P)$). Show that the set $\{t\in\mathcal{L}(P) \mid \text{all members of } I \text{ believe } t\}$ is consistent and deductively closed. Is the set $\{t \mid \text{over half the members of } I \text{ believe } t\}$ deductively closed or consistent?

7. A group G is called *orderable* if there exists a total ordering $\le$ on G such that $g\le h$ implies $gk\le hk$ and $kg\le kh$ for all k. Write down a propositional theory whose models are orderings of a given group G, and use the Compactness Theorem to show that G is orderable if and only if all its finitely-generated subgroups are orderable. Using the structure theorem for finitely-generated abelian groups, deduce that an abelian group is orderable if and only if it is torsion-free (i.e. it has no non-identity elements of finite order).

+ **8**. Let P be a set of primitive propositions. By a *Heyting valuation* of P we mean a function $v\colon P\to H$ from P to (the underlying set of) a Heyting algebra H (see sheet 1, question 13, for the definition). We extend v to a function $\overline{v}\colon\mathcal{L}(P)\to H$ in the obvious way: that is, $\overline{v}(\bot)$ is taken to be the least element 0 of H, and $\overline{v}(s\Rightarrow t)$ is the Heyting implication $\overline{v}(s)\Rightarrow\overline{v}(t)$ in H. A formula t is said to be a *Heyting tautology* if $\overline{v}(t)=1$ for all Heyting valuations v (in arbitrary Heyting algebras H) of the primitive propositions involved in t.

(i) Verify that the axioms (K) and (S) are Heyting tautologies, and deduce that any formula which is provable in the propositional calculus without using the third axiom is a Heyting tautology.

(ii) Show that $(\bot\Rightarrow q)$ is a Heyting tautology.

(iii) By considering a suitable valuation $\{p,q\}\to T$ where T is a three-element chain, show that the formula of question 1(iii) is not a Heyting tautology.

(iv) Is the formula $(((p\Rightarrow q)\Rightarrow r)\Rightarrow(((q\Rightarrow p)\Rightarrow r)\Rightarrow r))$ a Heyting tautology?

9. A class $\mathcal{C}$ of Σ-structures (for some given signature Σ) is said to be (*finitely*) *axiomatizable* if there is a (finite) set $\mathbb{T}$ of sentences in the language over Σ whose models are exactly the members of $\mathcal{C}$. Show (a) that if an axiomatizable class contains arbitrarily large finite structures, it must contain an infinite structure, and (b) that if $\mathcal{C}$ is finitely axiomatizable, then so is the class of all Σ-structures which are not in $\mathcal{C}$. Using these ideas, determine which of the following classes are axiomatizable, and which are finitely axiomatizable.

(i) the class of finite groups;
(ii) the class of infinite groups;
(iii) the class of groups of order n, for a given natural number n;
(iv) the class of torsion groups (i.e. groups in which all elements have finite order);
(v) the class of torsion-free groups (cf. question 7).

10. Show that the sentences $(\forall x, y)((x = y) \Rightarrow (y = x))$ and $(\forall x, y, z)((x = y) \Rightarrow ((y = z) \Rightarrow (x = z)))$ are theorems of the predicate calculus with equality. [There is no need to write out formal proofs in full; but you shouldn't expect your supervisor to be satisfied with an argument based on the Completeness Theorem (further exercise: why not?).]

11. By a *substructure* of an (Ω, Π)-structure A, we mean a subset B of the underlying set of A which is closed under the operations in Ω (that is, $b_1 \ldots, b_n \in B$ implies $\omega_A(b_1, \ldots, b_n) \in B$ for each $\omega \in \Omega$), made into an (Ω, Π)-structure by taking ω_B to be the restriction of ω_A to B^n, and $[\![\pi]\!]_B$ to be $[\![\pi]\!]_A \cap B^n$ for each $\pi \in \Pi$.

(i) Show that if B is a substructure of A and ϕ is a quantifier-free formula of $\mathcal{L}(\Omega, \Pi)$ (with n free variables, say), then $[\![\phi]\!]_B = [\![\phi]\!]_A \cap B^n$. Give an example to show that this equality may fail if ϕ contains quantifiers.

(ii) A first-order theory $\mathbb{T}$ is called *universal* if its axioms all have the form $(\forall \vec{x})\phi$ where $\vec{x}$ is a (possibly empty) string of variables and ϕ is quantifier-free. Show that if $\mathbb{T}$ is universal, then every substructure of a $\mathbb{T}$-model is a $\mathbb{T}$-model.

(iii) Similarly, $\mathbb{T}$ is called *inductive* if its axioms have the form $(\forall \vec{x})(\exists \vec{y})\phi$ where ϕ is quantifier-free. Show that if $\mathbb{T}$ is inductive, and A is a structure for the appropriate signature, then the set of substructures of A which are $\mathbb{T}$-models is closed under unions of chains.

(iv) For each of the following classes of structures, either write down a universal axiomatization of the class or show using (ii) that no such axiomatization exists. For those which are not universal, either write down an inductive axiomatization or show that no such axiomatization exists:

(a) the class of integral domains;
(b) the class of fields;
(c) the class of algebraically closed fields;
(d) the class of totally ordered sets;
(e) the class of ordered sets in which every element lies below some maximal element.

\+ **12**. Let $\mathbb{T}$ be a first-order theory over a signature Σ, and let $\mathbb{T}_\forall$ denote the set of all universal sentences (that is, sentences of the form in question 11(ii)) over Σ which are derivable from $\mathbb{T}$. Let M be a model of $\mathbb{T}_\forall$. Let Σ' be the signature obtained from Σ by adjoining one new constant c_m for each element m of M, and let $\mathbb{D}_M$ be the theory over Σ' consisting of all sentences $\phi[c_{m_1}, \ldots, c_{m_k}/x_1, \ldots, x_k]$ where ϕ is a quantifier-free formula over Σ with free variables $x_1, \ldots, x_k$, and $(m_1, \ldots, m_k) \in [\![\phi]\!]_M$. Show that $\mathbb{T} \cup \mathbb{D}_M$ is consistent, and deduce that there is a $\mathbb{T}$-model $\widehat{M}$ having a substructure isomorphic to M. Hence show that the converse of question 11(ii) holds, in the sense that if $\mathbb{T}$ is a first-order theory for which every substructure of a $\mathbb{T}$-model is a $\mathbb{T}$-model, then there is a universal theory (over the same signature) having the same models as $\mathbb{T}$.

[Method: suppose $\mathbb{T} \cup \mathbb{D}_M \vdash \bot$. Let F be a finite subset of this theory which is inconsistent; let ψ be the conjunction of the members of $\mathbb{D}_M$ which occur in F, and suppose $c_{m_1}, \ldots, c_{m_r}$ are the constants which appear in ψ. Let ψ' be the formula obtained from ψ on replacing the c_{m_i} by free variables x_i; show that $(\forall x_1, \ldots, x_r)\neg\psi'$ is derivable from $\mathbb{T}$, but not satisfied in M.]

13. Write down an axiomatization of the class of dense totally ordered sets (that is, totally ordered sets in which between any two elements there is a third one) without greatest or least members. Show that every countable model of this theory is isomorphic to the ordered set of rational numbers. Is every countable model of first-order Peano arithmetic isomorphic to the set of natural numbers? [Hint: compactness.]