Logic and Set Theory Examples 1

PTJ Lent 2016

Important Note: There are probably more questions on this example sheet than any one student will wish to attempt, and they are not intended to be all of the same level of difficulty. Some (marked by − signs) are mere five-finger exercises to ensure that you have understood the basic definitions, and may safely be omitted if you are **confident** that you have understood them. Others (marked by + signs) are more challenging and/or slightly off the syllabus, and are intended for students who wish to explore the subject a bit more widely than can be done in the course.

Comments on and/or corrections to the questions on this sheet are always welcome, and may be e-mailed to me at `ptj@dpmms.cam.ac.uk`.

− **1**. Write down all possible Hasse diagrams for a poset with four elements. [There are 16 of them.] How many of them are complete?

− **2**. Which of the following posets are complete? And which are chain-complete?

(i) The set of all finite subsets of an arbitrary set, ordered by inclusion.

(ii) The set of cofinite subsets (that is, subsets with finite complements) of an arbitrary set, ordered by inclusion.

(iii) The set of all transitive relations on an arbitrary set A, regarded as subsets of $A \times A$ and ordered by inclusion.

(iv) The set of all partial orderings of A, ordered by inclusion.

3. Let P and Q be posets. There are (at least) two possible ways of defining a partial ordering on $P \times Q$: the *pointwise order* is defined by $(a, c) \leq (b, d)$ if and only if $a \leq b$ and $c \leq d$, and the *lexicographic order* by $(a, c) \leq (b, d)$ if and only if either $a < b$ or ($a = b$ and $c \leq d$). Verify that both of these are partial orders. For each of the following properties, determine whether $P \times Q$ has the property (a) in the pointwise ordering, and (b) in the lexicographic ordering, if both P and Q have the property:

(i) being complete;

(ii) being totally ordered;

(iii) being chain-complete.

4. Let P be a complete poset and $f \colon P \to P$ an order-reversing map (i.e. one satisfying $(x \leq y) \Rightarrow (f(y) \leq f(x))$. Give an example to show that f need not have a fixed point. Show, however, that we can find a pair of (not necessarily distinct) points x, y with $f(x) = y$ and $f(y) = x$. Show further that we can find such a pair with $x \leq y$.

− **5**. For each of the following functions $\Phi \colon [\mathbb{N} \rightharpoonup \mathbb{N}] \to [\mathbb{N} \rightharpoonup \mathbb{N}]$, determine (a) whether Φ is order-preserving, and (b) whether it has a fixed point:

(i) $\Phi(f)(n) = f(n) + 1$ if $f(n)$ is defined, undefined otherwise.

(ii) $\Phi(f)(n) = f(n) + 1$ if $f(n)$ is defined, $\Phi(f)(n) = 0$ otherwise.

(iii) $\Phi(f)(n) = f(n-1) + 1$ if $f(n-1)$ is defined, $\Phi(f)(n) = 0$ otherwise.

6. Let P be a chain-complete poset with a least element, and $f \colon P \to P$ an order-preserving map. Show that the set P_f of fixed points of f has a least element and is chain-complete in the ordering it inherits from P. [Warning: the join of a chain of fixed points in P_f need not coincide with its join in P.] Deduce that if $f_1, f_2, \ldots, f_n$ are order-preserving maps $P \to P$ which commute with each other (i.e. $f_i \circ f_j = f_j \circ f_i$ for all i, j), then they have a common fixed point. Show by an example that two order-preserving maps $P \to P$ which do not commute with each other need not have a common fixed point.

7. We call a poset P *Bourbakian* if every order-preserving map $f \colon P \to P$ has a least fixed point $\mu(f)$. If P is Bourbakian, show that $\downarrow(p) = \{x \in P \mid x \leq p\}$ is Bourbakian for any $p \in P$ [hint: given an order-preserving $f : \downarrow(p) \to \downarrow(p)$, extend it to a map $P \to P$ by setting $f(y) = p$ for all $y \not\leq p$], and deduce that if $f, g \colon P \rightrightarrows P$ are order-preserving maps such that $f(x) \leq g(x)$ for all $x \in P$, then $\mu(f) \leq \mu(g)$. [Consider the restriction of f to $\downarrow(\mu(g))$.]

Now let P and Q be Bourbakian posets, and let $h\colon P \times Q \to P \times Q$ be a map which is order-preserving with respect to the pointwise ordering (cf. question 3). We denote the two components of the ordered pair $h(x, y)$ by $h_1(x, y)$ and $h_2(x, y)$ respectively. Show that, for each fixed $x \in P$, the mapping $g_x\colon Q \to Q$ defined by $g_x(y) = h_2(x, y)$ is order-preserving, and that $x \mapsto \mu(g_x)$ is an order-preserving map $P \to Q$. Hence show that the map $f\colon P \to P$ defined by $f(x) = h_1(x, \mu(g_x))$ is order-preserving, and that $(\mu(f), \mu(g_{\mu(f)}))$ is the least fixed point of h. [Thus a product of two Bourbakian posets is Bourbakian.]

8. A poset $(P, \leq)$ is said to be *inductive* if each chain in P has an upper bound (but not necessarily a least upper bound). The usual statement of Zorn's Lemma says that every inductive poset (and not merely every chain-complete poset) has enough maximal elements.

(i) Give an example of a poset which is inductive but not chain-complete.

(ii) If $(P, \leq)$ is any poset, let C denote the set of all chains in P, ordered by inclusion. Show that C is chain-complete.

(iii) If M is a maximal element of the poset C just defined, show that any upper bound for M in P is (a) a member of M, and (b) a maximal element of P.

(iv) Deduce the usual statement of Zorn's Lemma from the version proved in lectures.

9. Use Zorn's Lemma to prove

(i) that every partial ordering on a set can be extended to a total ordering;

(ii) that, for any two sets A and B, there exists either an injection $A \to B$ or an injection $B \to A$ [hint: consider the set of bijections from subsets of A to subsets of B, with a suitable ordering].

− **10**. Which of the following posets are lattices? Of those that are lattices, which are distributive?

(i) The set of all subspaces of a vector space, ordered by inclusion.

(ii) The set of natural numbers, ordered by divisibility.

(iii) The set $\Sigma*$ of all words over an alphabet Σ (that is, finite strings of members of Σ), ordered by the relation 'is a prefix of' (that is, $w \leq x$ iff $x = wz$ for some z).

(iv) The unit square $[0, 1] \times [0, 1]$ with the lexicographic ordering (cf. question 3).

− **11**. (i) Let L be the (five-element) lattice of subgroups of the non-cyclic group of order four. Show that there are no lattice homomorphisms $L \to 2$.

(ii) Find an example of a finite lattice L which admits at least one homomorphism $L \to 2$, but does not have enough such homomorphisms to separate its elements. [There is one with five elements.]

+ **12**. A *Boolean ring* is a ring (with 1) in which every element x satisfies $x^2 = x$.

(i) Show that in a Boolean ring R we have $x + x = 0$ and $xy = yx$ for all $x, y \in R$.

(ii) Show that, if we define a relation $\leq$ on R by setting $x \leq y$ if and only if $xy = x$, then $\leq$ is a partial order with respect to which R is a Boolean algebra.

(iii) Show that any Boolean algebra has the structure of a Boolean ring if we define multiplication to be binary meet and addition to be 'symmetric difference' (that is, $x + y = (x \wedge \neg y) \vee (\neg x \wedge y)$).

(iv) Show that if M is any maximal ideal of a Boolean ring R, then the quotient ring R/M is isomorphic to the field of integers mod 2. Hence show that the Maximal Ideal Theorem for rings can be used to give a proof of the Birkhoff–Stone Theorem for Boolean algebras.

+ **13**. A lattice L is called a *Heyting algebra* if, for any two elements $a, b \in L$, there is a unique largest $c \in L$ with $c \wedge a \leq b$. (The largest such c is usually denoted $a \Rightarrow b$; this defines a binary operation $\Rightarrow$ on L, which is called *Heyting implication*.)

(i) Show that every Boolean algebra is a Heyting algebra.

(ii) Show that every Heyting algebra is a distributive lattice.

(iii) Show that a complete lattice L is a Heyting algebra if and only if it satisfies the 'infinite distributive law'

$$a \wedge \bigvee S = \bigvee \{a \wedge s \mid s \in S\}$$

for all $a \in L$ and $S \subseteq L$. [Hint: consider $\bigvee\{c \in L \mid c \wedge a \leq b\}$.] Deduce that every finite distributive lattice is a Heyting algebra, and that the lattice $\mathcal{O}(X)$ of open subsets of any topological space X is a Heyting algebra.