Lent 2015 **LOGIC AND SET THEORY – EXAMPLES 2** IBL

1. Write down subsets of the reals that have order-types $\omega + \omega$, ω^2 and ω^3.

2. Let α and β be non-zero ordinals. Must we have $\alpha + \beta > \alpha$? Must we have $\alpha + \beta > \beta$?

3. Is there a non-zero ordinal α with $\alpha\omega = \alpha$? What about $\omega\alpha = \alpha$?

4. Show that the inductive and the synthetic definitions of ordinal multiplication coincide.

5. Let α, β, γ be ordinals. Prove that $(\alpha\beta)\gamma = \alpha(\beta\gamma)$.

6. Let α, β, γ be ordinals. Must we have $(\alpha + \beta)\gamma = \alpha\gamma + \beta\gamma$? Must we have $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$?

7. Let α and β be ordinals with $\alpha \geq \beta$. Show that there is a unique ordinal γ such that $\beta + \gamma = \alpha$. Must there exist an ordinal γ with $\gamma + \beta = \alpha$?

8. An ordinal written as $\omega^{\alpha_1} n_1 + \ldots + \omega^{\alpha_k} n_k$, where $\alpha_1 > \ldots > \alpha_k$ are ordinals (and k and $n_1, \ldots, n_k$ are non-zero natural numbers), is said to be in *Cantor Normal Form*. Show that every non-zero ordinal has a unique Cantor Normal Form. What is the Cantor Normal Form for the ordinal ϵ_0?

9. Is ω_1 a successor or a limit?

10. Let α be a countable (non-zero) limit ordinal. Prove that there exists an increasing sequence $\alpha_1 < \alpha_2 < \alpha_3 < \ldots$ with supremum equal to α. Is this result true for $\alpha = \omega_1$?

11. Show that, for every countable ordinal α, there is a subset of $\mathbb{Q}$ of order-type α. Why is there no subset of $\mathbb{R}$ of order-type ω_1?

12. An operation $\alpha * \beta$ is defined on ordinals as follows. We set $0 * \beta$ to be ω^β, and $\alpha^+ * \beta$ to be the β-th ordinal γ such that $\alpha * \gamma = \gamma$, and $\lambda * \beta$ to be the supremum of the set $\{\alpha * \beta : \alpha < \lambda\}$ for λ a non-zero limit. Explain why this definition makes sense. Describe the ordinals $1 * 0$, $1 * 1$, $1 * 2$ and $2 * 0$. Is there a countable ordinal α such that $\alpha * 0 = \alpha$?

13. Is it possible to select for each countable (non-zero) limit ordinal α an ordinal $x_\alpha < \alpha$ in such a way that the x_α are distinct?

$^+$14. Let X be a totally ordered set such that the only order-preserving injection from X to itself is the identity. Must X be finite?