Logic and Set Theory Examples 3

PTJ Michaelmas 2012

Comments on and/or corrections to the questions on this sheet are always welcome, and may be e-mailed to me at `ptj@dpmms.cam.ac.uk`.

1. Is the collection of sets z satisfying

$$\neg(\exists u_1, \ldots, u_n)((z \in u_1) \wedge (u_1 \in u_2) \wedge \cdots \wedge (u_n \in z))$$

a set for any n? [Try to answer this **without** using the axiom of Foundation.]

− **2**. Show that the Pair-Set axiom is deducible from the axioms of Empty-Set, Power-Set and Replacement.

− **3**. Show that if x is a transitive set, then $\mathcal{P}x$ and $\bigcup x$ are also transitive. Are the converses true?

4. Use $\in$-induction to prove that the only automorphism of the structure $(V, \in)$ which is definable by a function-class is the identity.

5. Let $(V, \in)$ be a model of ZF, and let σ be a permutation of V (which you may assume to be given by a function-class). We define a new binary relation $\in^\sigma$ on V by $(x \in^\sigma y) \Leftrightarrow (x \in \sigma(y))$.

(i) Verify that the structure $(V, \in^\sigma)$ satisfies all the axioms of ZF except possibly for Foundation.

(ii) By taking σ to be the transposition which interchanges $\emptyset$ and $\{\emptyset\}$ (and fixes everything else), show that Foundation may fail.

(iii) More generally, let a be a set none of whose members is a singleton, and let σ be the permutation which interchanges x and $\{x\}$ for each $x \in a$. Show that $(V, \in^\sigma)$ satisfies a weak version of Foundation which says that every nonempty set x has a member y satisfying either $x \cap y = \emptyset$ or $y = \{y\}$.

6. Define S to be the smallest set a such that $(\emptyset \in a) \wedge (\forall x, y \in a)(x \cup \{y\} \in a)$.

(i) Explain how the axioms of ZF ensure that such a set exists and is unique.

(ii) Show that S is closed under $\cup$ (that is, $(\forall x, y \in S)((x \cup y) \in S)$).

(iii) Show that S is closed under $\bigcup$.

(iv) Show that S is the smallest set containing $\emptyset$ and closed under taking pairs and unions.

7. Use the $\in$-recursion theorem to show that there is a unique function-class $\overline{\mathrm{TC}}$ such that

$$(\forall x)(\overline{\mathrm{TC}}(x) = x \cup \bigcup\{\overline{\mathrm{TC}}(y) \mid y \in x\}) ,$$

and show that $\overline{\mathrm{TC}}$ coincides with the transitive closure operation as defined in lectures. Why is $\overline{\mathrm{TC}}$ unsatisfactory as a definition of transitive closure?

8. A class M is *transitive* if $(\forall x, y)(((x \in y) \wedge (y \in M)) \Rightarrow (x \in M))$ holds. Show that if M is a transitive class, then the structure $(M, \in)$ satisfies the axiom of extensionality, and that it satisfies each of the empty-set, pair-set and union-set axioms if and only if M is closed under the corresponding finitary operation on V. What more do you need to know about M to get a similar result for the power-set axiom?

9. If P is a property of sets, a set x is said to be *hereditarily* P if every member of $\mathrm{TC}(\{x\})$ has property P. Consider the classes HF, HC and HS of hereditarily finite, hereditarily countable and hereditarily small sets (where we call a set *small* if it can be injected into one of the sets $\omega, \mathcal{P}\omega, \mathcal{P}\mathcal{P}\omega, \ldots$): in each case determine which axioms of ZF hold, and which fail, in the structure obtained from the class as in the previous question.

10. Prove that each of the following is an alternative characterization of the set V_ω (the ωth stage in the von Neumann hierarchy):

(i) V_ω is the class HF of hereditarily finite sets (cf. question 9);

(ii) V_ω is the class $\{x \mid \mathrm{TC}(\{x\}) \text{ is finite}\}$ of *strongly hereditarily finite* sets;

(iii) V_ω is the set S considered in question 6;

(iii) V_ω is the smallest set containing $\emptyset$ and closed under $\mathcal{P}$ and under formation of arbitrary subsets.

Deduce in particular that the class HF is a set. Is HC a set? If so, does it coincide with V_α for any α?

11. Consider the binary relation E on $\mathbb{N}$ defined by: $n\,E\,m$ iff the $(n+1)$st digit (counting from the right) in the binary expansion of m is 1. What can you say about the structure $(\mathbb{N}, E)$?

+ **12**. Let $(V, \in)$ be a structure satisfying all the axioms of ZF except Foundation, and also satisfying the weak Foundation axiom of question 4(iii). Suppose also that the *autosingletons* of V (that is, the elements x satisfying $x = \{x\}$) form an infinite set a.

(i) Show that we may define a 'rank function' $V \to \mathbf{On}$ such that each autosingleton has rank 0, and every other set x satisfies $\mathrm{rank}(x) = \bigcup\{\mathrm{rank}(y)^+ \mid y \in x\}$.

(ii) Let G be the group of all permutations of a; show that we may extend each $\pi \in G$ to a permutation π^* of V satisfying $\pi^*(x) = \{\pi^*(y) \mid y \in x\}$ for all x.

(iii) We define a set x to be *of finite support* if there exists a finite set $a' \subseteq a$ such that $\pi^*(x) = x$ whenever π fixes all the members of a'. Show that the class HFS of members of V which are hereditarily of finite support (in the sense defined in question 8) is a model of ZF minus Foundation. Show also that $a \in HFS$, but that

$$HFS \models \neg(\exists b \subseteq a \times a)(b \text{ is a total ordering of } a) \ .$$

(iv) Now suppose a has a dense total ordering without endpoints [if you wish, take $a \cong \mathbb{Q}$], and replace the group of all permutations of a by the subgroup H of order-preserving permutations. Defining the notions of (hereditary) finite support as before, but restricting to permutations in H, show that HFS is still a model of ZF minus Foundation, and that the given total ordering of a belongs to it. Show also that the set I of all *intervals* $(x, y) = \{z \in a \mid x < z < y\}$ with $x < y$ in a belongs to HFS, but that there is no function $f \colon I \to a$ in HFS with $f(i) \in i$ for all $i \in I$.

13. (i) Determine the rank of the set $\mathbb{R}$ of real numbers. [You may assume that a real number is an ordered pair of subsets of $\mathbb{Q}$ (a Dedekind section), that a rational number is an equivalence class of ordered pairs of integers, and so on.]

(ii) Show that there is a subset of $\mathbb{R}$ which (under the restriction of the usual ordering on $\mathbb{R}$) is order-isomorphic to $\omega + \omega$.

(iii) Show that all the axioms of ZF except for the scheme of Replacement hold in $V_{\omega+\omega}$. Why can we deduce from (ii) that Replacement does **not** hold?

14. Let r be a binary relation on a set a. Show that r is well-founded iff there exists a function $h \colon a \to \alpha$ for some ordinal α, such that $\langle x, y\rangle \in r$ implies $h(x) < h(y)$. Deduce that if every set can be well-ordered, then any well-founded binary relation on a set can be extended to a well-ordering. [Compare question 9(i) on sheet 1.]