Logic and Set Theory Examples 2

PTJ Michaelmas 2012

Comments on and/or corrections to the questions on this sheet are always welcome, and may be e-mailed to me at `ptj@dpmms.cam.ac.uk`.

− **1**. Which of the following propositional formulae are tautologies?
(i) $((p \Rightarrow (q \Rightarrow r)) \Rightarrow (q \Rightarrow (p \Rightarrow r)))$;
(ii) $(((p \Rightarrow q) \Rightarrow r) \Rightarrow ((q \Rightarrow p) \Rightarrow r))$;
(iii) $(((p \Rightarrow q) \Rightarrow p) \Rightarrow p)$;
(iv) $((p \Rightarrow (p \Rightarrow q)) \Rightarrow p)$.

− **2**. Use the Deduction Theorem to show that the converse of the third axiom (i.e. the formula $(p \Rightarrow \neg\neg p))$ is a theorem of the propositional calculus.

3. Let t be a propositional formula not involving the constant $\perp$, and let $t' = t[\perp/p]$ be the formula obtained from t by substituting $\perp$ for all occurrences of a particular propositional variable p in t. Suppose that t' is a tautology but t is not; show that any proof of t' in the propositional calculus must involve an instance of the third axiom. Does this result remain true (a) if t is allowed to contain occurrences of $\perp$, or (b) if $\perp$ is replaced by $\top$?

4. Write down a proof of $(\perp \Rightarrow q)$ in the propositional calculus [hint: observe the result of question 3], and hence write down a deduction of $(p \Rightarrow q)$ from $\{\neg p\}$.

5. Show that if there is a deduction of t from $S \cup \{s\}$ in n lines (that is, consisting of n consecutive formulae), then $(s \Rightarrow t)$ is deducible from S in at most $3n + 2$ lines. Show further that there is a deduction of $\perp$ from $\{((p \Rightarrow q) \Rightarrow p), (p \Rightarrow \perp)\}$ in 16 lines [hint: use question 4], and hence calculate an upper bound for the length of a proof of the tautology of question 1(iii).

6. The beliefs of each member of a finite non-empty set I of individuals are represented by a consistent, deductively closed set of propositional formulae (in some fixed language $\mathcal{L}(P)$). Show that the set $\{t \in \mathcal{L}(P) \mid \text{all members of } I \text{ believe } t\}$ is consistent and deductively closed. Is the set $\{t \mid \text{over half the members of } I \text{ believe } t\}$ deductively closed or consistent?

7. A group G is called *orderable* if there exists a total ordering $\leq$ on G such that $g \leq h$ implies $gk \leq hk$ and $kg \leq kh$ for all k. Write down a propositional theory whose models are orderings of a given group G, and use the Compactness Theorem to show that G is orderable if and only if all its finitely-generated subgroups are orderable. Using the structure theorem for finitely-generated abelian groups, deduce that an abelian group is orderable if and only if it is torsion-free (i.e. it has no non-identity elements of finite order).

+ **8**. Let P be a set of primitive propositions. By a *Heyting valuation* of P we mean a function $v : P \to H$ from P to (the underlying set of) a Heyting algebra H (see sheet 1, question 12, for the definition). We extend v to a function $\overline{v} : \mathcal{L}(P) \to H$ in the obvious way: that is, $\overline{v}(\perp)$ is taken to be the least element 0 of H, and $\overline{v}(s \Rightarrow t)$ is the Heyting implication $\overline{v}(s) \Rightarrow \overline{v}(t)$ in H. A formula t is said to be a *Heyting tautology* if $\overline{v}(t) = 1$ for all Heyting valuations v (in arbitrary Heyting algebras H) of the primitive propositions involved in t.

(i) Verify that the axioms (K) and (S) are Heyting tautologies, and deduce that any formula which is provable in the propositional calculus without using the third axiom is a Heyting tautology.

(ii) Show that $(\perp \Rightarrow q)$ is a Heyting tautology.

(iii) By considering a suitable valuation $\{p, q\} \to T$ where T is a three-element chain, show that the formula of question 1(iii) is not a Heyting tautology.

(iv) Is the formula $(((p \Rightarrow q) \Rightarrow r) \Rightarrow (((q \Rightarrow p) \Rightarrow r) \Rightarrow r))$ a Heyting tautology?

9. Describe sets of axioms in suitable first-order languages (to be specified) for the following theories:

(i) the theory of ordered groups (i.e., groups with a compatible total ordering, cf. question 7);

(ii) the theory of fields;

(iii) the theory of fields of order p, for a fixed prime p [hint: take a signature with plenty of constants];

(iv) the theory of infinite-dimensional vector spaces over a (given) finite field F [hint: first express the assertion '$\{x_1, \ldots, x_n\}$ is linearly independent' as a finite conjunction];

(v) the theory of algebraically closed fields of characteristic zero;

(vi) the theory of posets in which every element lies below some maximal element;

(vii) the theory of totally ordered sets which are densely ordered (i.e. between any two elements there lies a third one) and have neither greatest nor least elements.

10. By a *substructure* of an (Ω, Π)-structure A, we mean a subset B of the underlying set of A which is closed under the operations in Ω (that is, $b_1 \ldots, b_n \in B$ implies $\omega_A(b_1, \ldots, b_n) \in B$ for each $\omega \in \Omega$), made into an (Ω, Π)-structure by taking ω_B to be the restriction of ω_A to B^n, and $[\pi]_B$ to be $[\pi]_A \cap B^n$ for each $\pi \in \Pi$.

(i) Show that if B is a substructure of A and ϕ is a quantifier-free formula of $\mathcal{L}(\Omega, \Pi)$ (with n free variables, say), then $[\phi]_B = [\phi]_A \cap B^n$. Give an example to show that this equality may fail if ϕ contains quantifiers.

(ii) A first-order theory T is called *universal* if its axioms all have the form $(\forall \vec{x})\phi$ where $\vec{x}$ is a (possibly empty) string of variables and ϕ is quantifier-free. Show that if T is universal, then every substructure of a T-model is a T-model.

(iii) Similarly, T is called *inductive* if its axioms have the form $(\forall \vec{x})(\exists \vec{y})\phi$ where ϕ is quantifier-free. Show that if T is inductive, and A is an (Ω, Π)-structure, then the set of substructures of A which are T-models is closed under unions of chains.

(iv) Which of the theories of question 9 are (axiomatizable as) universal theories? And which are inductive?

\+ **11**. Let T be a first-order theory over a signature Σ, and let $T_\forall$ denote the set of all universal sentences over Σ which are derivable from T. Let M be a model of $T_\forall$. Let Σ' be the signature obtained from Σ by adjoining one new constant c_m for each element m of M, and let D_M be the theory over Σ' consisting of all sentences $\phi[c_{m_1}, \ldots, c_{m_k}/x_1, \ldots, x_k]$ where ϕ is a quantifier-free formula over Σ with free variables $x_1, \ldots, x_k$, and $(m_1, \ldots, m_k) \in [\phi]_M$. Show that $T \cup D_M$ is consistent, and deduce that there is a T-model $\widehat{M}$ having a substructure isomorphic to M. Hence show that the converse of question 10(ii) holds, i.e. that if T is a first-order theory for which every substructure of a T-model is a T-model, then there is a universal theory (over the same signature) having the same models as T.

[Method: suppose $T \cup D_M \vdash \bot$. Let F be a finite subset of this theory which is inconsistent; let ψ be the conjunction of the members of D_M which occur in F, and suppose $c_{m_1}, \ldots, c_{m_r}$ are the constants which appear in ψ. Let ψ' be the formula obtained from ψ on replacing the c_{m_i} by variables x_i; show that $(\forall x_1, \ldots, x_r)\neg\psi'$ is derivable from T, but not satisfied in M.]

12. Show that the sentences $(\forall x, y)((x = y) \Rightarrow (y = x))$ and $(\forall x, y, z)((x = y) \Rightarrow ((y = z) \Rightarrow (x = z)))$ are theorems of the predicate calculus with equality. [There is no need to write out formal proofs in full; but you shouldn't expect your supervisor to be satisfied with an argument based on the Completeness Theorem (further exercise: why not?).]

13. Show that every countable model of the theory of question 9(vii) is isomorphic to the ordered set of rational numbers. Is every countable model of first-order Peano arithmetic isomorphic to the set of natural numbers? [Hint: compactness.]