Logic and Set Theory Examples 4

PTJ Lent 2012

Comments on and/or corrections to the questions on this sheet are always welcome, and may be e-mailed to me at `ptj@dpmms.cam.ac.uk`.

− **1**. Prove or disprove the following statements:
(i) $\alpha + \beta$ is a limit ordinal if and only if β is a limit ordinal.
(ii) $\alpha \,.\, \beta$ is a limit ordinal if and only if either α or β is a limit ordinal.
(iii) Any limit ordinal can be written in the form $\omega \,.\, \alpha$ for some α.
(iv) Any limit ordinal can be written in the form $\alpha \,.\, \omega$ for some α.

2. Ordinal subtraction is defined synthetically by taking $\alpha - \beta$ to be the order-type of the set-difference $\alpha \setminus \beta$ (in particular, $\alpha - \beta = 0$ whenever $\alpha \leq \beta$). Prove the following identities:

$$(\alpha + \beta) - \alpha = \beta \quad ; \quad \alpha - (\beta + \gamma) = (\alpha - \beta) - \gamma \quad ; \quad \alpha \,.\, (\beta - \gamma) = \alpha \,.\, \beta - \alpha \,.\, \gamma \quad .$$

Show also that for any ordinal α there are only finitely many ordinals of the form $\alpha - \beta$. [Hint: consider the order-type of the set $\{\alpha - \beta \mid \beta \in \mathbf{On}\}$.]

3. A function-class $F \colon \mathbf{On} \to \mathbf{On}$ is called a *normal function* if it is strictly order-preserving (i.e. $\alpha < \beta$ implies $F(\alpha) < F(\beta)$) and continuous at limits (i.e. $F(\lambda) = \bigcup\{F(\alpha) \mid \alpha < \lambda\}$ for nonzero limits λ). Prove the following facts about a normal function F:
(i) $F(\alpha) \geq \alpha$ for all α;
(ii) $\bigcup\{0, F(0), F(F(0)), \ldots\}$ is the least ordinal β satisfying $F(\beta) = \beta$;
(iii) there is a normal function G whose range is exactly the class of ordinals β satisfying $F(\beta) = \beta$.

Find G when F is the function $\beta + (-)$ for some β, and when it is the function $\gamma \,.\, (-)$ for some nonzero γ.

4. Show that every ordinal α has a unique representation (its *Cantor Normal Form*) of the form

$$\alpha = \omega^{\alpha_1} \,.\, a_1 + \omega^{\alpha_2} \,.\, a_2 + \cdots + \omega^{\alpha_n} \,.\, a_n$$

where $n \in \omega$, $\alpha \geq \alpha_1 > \alpha_2 > \cdots > \alpha_n$, and $a_1, a_2, \ldots, a_n$ are nonzero natural numbers. [Hint: consider the least β such that $\omega^\beta > \alpha$; can it be a limit?] Describe how to calculate the Cantor Normal Form of $\alpha + \beta$ from those of α and β.

+ **5**. Consider the binary operation $\oplus$ on **On** defined recursively by setting

$$\alpha \oplus \beta = \bigcup(\{(\alpha \oplus \beta')^+ \mid \beta' < \beta\} \cup \{(\alpha' \oplus \beta)^+ \mid \alpha' < \alpha\}) \quad .$$

Prove the following facts about $\oplus$:
(i) $\oplus$ is commutative and associative, and has 0 as an identity element.
(ii) If $n < \omega$, then $\alpha \oplus n = \alpha + n$ for any α.
(iii) For any α, the set of ordinals less than ω^α is closed under $\oplus$.
(iv) If $\beta < \omega^\alpha$, then $\omega^\alpha \oplus \beta = \omega^\alpha + \beta$.
(v) $\omega^\alpha \,.\, m \oplus \omega^\alpha \,.\, n = \omega^\alpha \,.\, (m + n)$ for any $m, n < \omega$.
(vi) $\oplus$ coincides with the binary operation obtained by treating Cantor Normal Forms as if they were polynomials in ω, and applying the usual rules for adding polynomials.

[N.B.: You will need to prove (iii)–(vi) by a *simultaneous* induction.]

− **6**. Show that the assertion 'For any two sets a and b, there exists either an injection $a \to b$ or an injection $b \to a$' is equivalent to the axiom of choice. [One direction was done in question 8(ii) on sheet 1; for the converse, use Hartogs' Lemma.]

7. Let r be a binary relation on a set a. Show that r is well-founded iff there exists a function $h\colon a \to \alpha$ for some ordinal α, such that $\langle x, y\rangle \in r$ implies $h(x) < h(y)$. Deduce, using the axiom of choice, that any well-founded binary relation on a set can be extended to a well-ordering.

+ **8**. By a *selection function* for a set a, we mean a function $s\colon \mathcal{P}a \to \mathcal{P}a$ such that $\emptyset \subseteq s(b) \subseteq b$ for all $b \subseteq a$, both inclusions being strict unless b is empty or a singleton.

(i) Show that $\mathcal{P}\alpha$ has a selection function for any ordinal α. [Given a subset $b \subseteq \mathcal{P}\alpha$ with more than one element, consider the least β belonging to some but not all members of b.]

(ii) Conversely, suppose a has a selection function s. For any ordinal α and function $f : \alpha \to 2$, we define a subset $b(f)$ of a by the following recursion: if $\alpha = 0$, $b(f) = a$; if α is a limit, $b(f) = \bigcap\{b(f|_\beta) \mid \beta < \alpha\}$; if $\alpha = \beta^+$ and $f(\beta) = 1$, $b(f) = s(b(f|_\beta))$; if $\alpha = \beta^+$ and $f(\beta) = 0$, $b(f) = b(f|_\beta) \setminus s(b(f|_\beta))$. Show that, for each ordinal α, the nonempty members of the family $\{b(f) \mid \mathrm{dom}\ f = \alpha\}$ form a partition of a. Show also that $b(f)$ has at most one element for every f with domain $\gamma(\mathcal{P}a)$, and deduce that there is an injection from a to the power-set of an ordinal.

(iii) Deduce that the assertion 'Every set admits a selection function' implies that every set can be totally ordered.

9. (i) Show that the statement 'Every set can be totally ordered' implies that every family of nonempty finite sets has a choice function.

(ii) By a *multiple-choice function* for a set a, we mean a function picking out a nonempty finite subset of each nonempty subset of a. Show that the assertion 'Every set has a multiple-choice function' implies that any totally orderable set can be well-ordered.

10 For each countable ordinal α, show that there is a subset of $\mathbb{R}$ which is well-ordered (in the usual ordering) and has order-type α. Is there a well-ordered subset of $\mathbb{R}$ (again, in the usual ordering) of order-type ω_1?

11. Suppose the axiom of choice fails to the extent that no uncountable subset of $\mathbb{R}$ can be well-ordered (equivalently, the Hartogs ordinal $\gamma(\mathbb{R})$ is ω_1). Let a be the set of all binary relations on ω (that is, subsets of $\omega \times \omega$) and b the quotient of a by the equivalence relation which identifies two relations r and s iff there is a permutation π of ω such that $(m, n) \in r \Leftrightarrow (\pi(m), \pi(n)) \in s$ for all $m, n \in \omega$. By considering equivalence classes whose members are equivalence relations, or otherwise, show that there is an injection $a \to b$ (as well as the evident surjection $a \to b$). By considering equivalence classes whose members are well-orderings, show that there is no bijection $a \to b$.

12 A subset x of an ordinal α is said to be *cofinal* if, for every $\beta \in \alpha$, there exists $\gamma \in x$ with $\beta \leq \gamma$. We define the *cofinality* $\mathrm{cf}(\alpha)$ of α to be the least ordinal β for which there exists an order-preserving map $\beta \to \alpha$ whose image is cofinal in α. Prove that

(i) for any α, $\mathrm{cf}(\mathrm{cf}(\alpha)) = \mathrm{cf}(\alpha)$;

(ii) for any limit ordinal α, $\mathrm{cf}(\alpha)$ is an initial ordinal;

(iii) for any successor ordinal α, $\mathrm{cf}(\omega_\alpha) = \omega_\alpha$;

(iv) for a nonzero limit ordinal λ, we have either $\mathrm{cf}(\omega_\lambda) < \omega_\lambda$ or $\omega_\lambda = \lambda$.

Show that there is a least ordinal α such that $\omega_\alpha = \alpha$. What is its cofinality?

+ **13**. (i) Show that any ordinal α may be made into a topological space by taking all subsets of the forms $\{\delta \in \alpha \mid \beta < \delta\}$ and $\{\delta \in \alpha \mid \delta < \gamma\}$, and pairwise intersections of these, as a basis for the open sets.

(ii) Show that, in this topology, α is compact iff it is either zero or a successor ordinal.

(iii) Suppose that α is a limit and $\mathrm{cf}(\alpha) > \omega$ (cf. the previous question). Show that α has the 'Bolzano–Weierstrass property' that every sequence has a convergent subsequence [hint: recall a proof from Analysis I that every sequence has a monotonic subsequence]. Show also that every continuous function $f : \alpha \to \mathbb{R}$ is bounded. [First show that there exists $\beta < \alpha$ such that $|f(\beta) - f(\gamma)| < 1$ for all $\gamma > \beta$.]

(iv) Show that the topology on α can be induced by a metric if and only if $\alpha < \omega_1$. [For 'if', use the idea of question 10; for 'only if', use parts (ii) and (iii).]