Logic and Set Theory Examples 3

PTJ Lent 2012

Comments on and/or corrections to the questions on this sheet are always welcome, and may be e-mailed to me at `ptj@dpmms.cam.ac.uk`.

1. Show that the sentences $(\forall x, y)((x = y) \Rightarrow (y = x))$ and $(\forall x, y, z)((x = y) \Rightarrow ((y = z) \Rightarrow (x = z)))$ are theorems of the predicate calculus with equality. [There is no need to write out formal proofs in full; but you shouldn't expect your supervisor to be satisfied with an argument based on the Completeness Theorem (further exercise: why not?).]

2. Show that every countable model of the theory of densely totally ordered sets without greatest or least elements (cf. question 10(vii) on sheet 2) is isomorphic to the ordered set of rational numbers. Is every countable model of first-order Peano arithmetic isomorphic to the set of natural numbers?

\+ **3**. Let T be a first-order theory over a signature Σ.

(i) The *Lindenbaum algebra* of T is the set $\mathcal{B}(T)$ of equivalence classes of sentences in $\mathcal{L}(\Sigma)$ modulo the relation $\equiv_T$, where $\phi \equiv_T \psi$ if and only if both $(\phi \Rightarrow \psi)$ and $(\psi \Rightarrow \phi)$ are derivable from T. Explain how $\mathcal{B}(T)$ becomes a Boolean algebra when ordered by the relation $\leq_T$, where $[\phi] \leq_T [\psi]$ if and only if $T \vdash (\phi \Rightarrow \psi)$. [Detailed proofs are not required.]

(ii) We say that T has the *Ryll-Nardzewski property* if $\mathcal{B}_n(T)$ is finite for all n, where $\mathcal{B}_n(T)$ denotes the Lindenbaum algebra of T regarded as a theory over the signature obtained by adjoining n new constants $(c_1, \ldots, c_n)$ to Σ (but no new axioms). Show that if T has the Ryll-Nardzewski property, then for any n-tuple of elements $(a_1, \ldots, a_n)$ of a T-model A there exists a unique atom (that is, a minimal nonzero element) $[\phi]$ of $\mathcal{B}_n(T)$ such that A satisfies ϕ when we interpret the new constants c_i as a_i $(1 \leq i \leq n)$. [You may assume the result that any finite Boolean algebra is isomorphic to a power-set.]

(iii) Deduce that if T is also complete (that is, $\mathcal{B}_0(T)$ has just two elements), then any two (finite or) countable models of T are isomorphic. [Try to mimic the proof you gave for the first part of question 2.]

(iv) A Boolean algebra is said to be *atomless* if it is nontrivial (i.e. satisfies $0 \neq 1$) and has no atoms; equivalently, if any nonzero element can be written as the join of two strictly smaller elements. Write down a first-order axiomatization of the theory of atomless Boolean algebras, and show that it is complete and has the Ryll-Nardzewski property. [Hint: for any n, there are only finitely many isomorphism types of Boolean algebras that can be generated by n elements.] What is the unique countable model of this theory?

4. Is the collection of sets z satisfying

$$\neg(\exists u_1, \ldots, u_n)((z \in u_1) \wedge (u_1 \in u_2) \wedge \cdots \wedge (u_n \in z))$$

a set for any n? [Try to answer this **without** using the axiom of Foundation.]

− **5**. Show that the Pair-Set axiom is deducible from the axioms of Empty-Set, Power-Set and Replacement.

6. Use $\in$-induction to prove that the structure $(V, \in)$ has no automorphisms other than the identity.

7. Define S to be the smallest set a such that $(\emptyset \in a) \wedge (\forall x, y \in a)(x \cup \{y\} \in a)$.

(i) Explain how the axioms of ZF ensure that such a set exists and is unique.

(ii) Show that S is closed under $\cup$ (that is, $(\forall x, y \in S)((x \cup y) \in S)$).

(iii) Show that S is closed under $\bigcup$.

(iv) Show that S is the smallest set containing $\emptyset$ and closed under taking pairs and unions.

8. Let $(V, \in)$ be a model of ZF, and let $\varepsilon \subseteq V \times V$ be the relation defined by $x \,\varepsilon\, y \Leftrightarrow x \in \sigma(y)$, where σ is the permutation of V which interchanges $\emptyset$ and $\{\emptyset\}$ and fixes everything else. Verify that the structure (V, ε) satisfies all the axioms of ZF except Foundation, and that it contains an element x satisfying $x \,\varepsilon\, x$.

9. Use the $\in$-recursion theorem to show that there is a unique function-class $\overline{\mathrm{TC}}$ such that

$$(\forall x)(\overline{\mathrm{TC}}(x) = x \cup \bigcup\{\overline{\mathrm{TC}}(y) \mid y \in x\}) \ ,$$

and show that $\overline{\mathrm{TC}}$ coincides with the transitive closure operation as defined in lectures. Why is $\overline{\mathrm{TC}}$ unsatisfactory as a definition of transitive closure?

10. A class M is *transitive* if $(\forall x, y)(((x \in y) \wedge (y \in M)) \Rightarrow (x \in M))$ holds. Show that if M is a transitive class, then the structure $(M, \in)$ satisfies the axiom of extensionality, and that it satisfies each of the empty-set, pair-set and union-set axioms if and only if M is closed under the corresponding finitary operation on V. What more do you need to know about M to get a similar result for the power-set axiom?

11. If P is a property of sets, a set x is said to be *hereditarily* P if every member of $\mathrm{TC}(\{x\})$ has property P. Consider the classes HF, HC and HS of hereditarily finite, hereditarily countable and hereditarily small sets (where we call a set *small* if it can be injected into one of the sets $\omega, \mathcal{P}\omega, \mathcal{P}\mathcal{P}\omega, \ldots$): in each case determine which axioms of ZF hold, and which fail, in the structure obtained from the class as in the previous question.

12. Prove that each of the following is an alternative characterization of the set V_ω (the ωth stage in the von Neumann hierarchy):

(i) V_ω is the class HF of hereditarily finite sets (cf. question 12);

(ii) V_ω is the class $\{x \mid \mathrm{TC}(\{x\}) \text{ is finite}\}$ of *strongly hereditarily finite* sets;

(iii) V_ω is the set S considered in question 7;

(iii) V_ω is the smallest set containing $\emptyset$ and closed under $\mathcal{P}$ and under formation of arbitrary subsets.

Deduce in particular that the class HF is a set. Is HC a set? If so, does it coincide with V_α for any α?

13. Consider the binary relation E on $\mathbb{N}$ defined by: $n \,E\, m$ iff the $(n+1)$st digit (counting from the right) in the binary expansion of m is 1. What can you say about the structure $(\mathbb{N}, E)$?

14. (i) Determine the rank of the set $\mathbb{R}$ of real numbers. [You may assume that a real number is an ordered pair of subsets of $\mathbb{Q}$ (a Dedekind section), that a rational number is an equivalence class of ordered pairs of integers, and so on.]

(ii) Show that there is a subset of $\mathbb{R}$ which (under the restriction of the usual ordering on $\mathbb{R}$) is order-isomorphic to $\omega + \omega$.

(iii) Show that all the axioms of ZF except for the scheme of Replacement hold in $V_{\omega+\omega}$. Why can we deduce from (ii) that Replacement does **not** hold?