Lent 2010 **LOGIC AND SET THEORY – EXAMPLES 1** IBL

1. Which of the following propositions are tautologies?
(i) $(p_1 \Rightarrow (p_2 \Rightarrow p_3)) \Rightarrow (p_2 \Rightarrow (p_1 \Rightarrow p_3))$
(ii) $((p_1 \vee p_2) \wedge (p_1 \vee p_3)) \Rightarrow (p_2 \vee p_3)$
(iii) $(p_1 \Rightarrow (\neg p_2)) \Rightarrow (p_2 \Rightarrow (\neg p_1))$

2. Write down a proof of $\perp \Rightarrow q$. Use this to write down a proof of $p \Rightarrow q$ from $\neg p$.

3. Use the Deduction Theorem to show that $p \vdash \neg\neg p$.

4. Show that $\{p, q\} \vdash p \wedge q$ in three different ways: by writing down a proof, by using the Deduction Theorem, and by using the Completeness Theorem.

5. Give propositions p and q for which $(p \Rightarrow q) \Rightarrow \neg(q \Rightarrow p)$ is a tautology.

6. Explain carefully why the set of all propositions is countable.

7. Three people each have a set of beliefs: a consistent deductively closed set. Show that the set of propositions that they all believe is also consistent and deductively closed. Must the set of propositions that a majority believe be consistent? Must it be deductively closed?

8. Can the third axiom be deduced from the first two? In other words, is there a proof of $(\neg\neg p) \Rightarrow p$ that uses only the first two axioms and modus ponens?

9. Let $t_1, t_2, \ldots$ be propositions such that, for every valuation v, there exists n with $v(t_n) = 1$. Use the Compactness Theorem to show that in fact we may bound the values of n: there must be an N such that, for every valuation v, there exists $n \leq N$ with $v(t_n) = 1$.

10. Two sets S, T of propositions are *equivalent* if $S \vdash t$ for every $t \in T$ and $T \vdash s$ for every $s \in S$. A set S of propositions is *independent* if for every $s \in S$ we have $S - \{s\} \not\vdash s$. Show that every finite set of propositions has an independent subset equivalent to it. Give an infinite set of propositions that has no independent subset equivalent to it. Show, however, that for every set of propositions there exists an independent set equivalent to it.

11. Give a direct proof of the Compactness Theorem (not making use of the notion of syntactic implication).

12. Give an explicit function f from natural numbers to natural numbers such that every tautology of length n has a proof that is at most $f(n)$ lines long.

13. A set S of propositions is a *chain* if for any distinct $p, q \in S$ we have $p \vdash q$ or $q \vdash p$ but not both. Write down an infinite chain. If the set of primitive propositions is allowed to be uncountable, can there exist an uncountable chain?

$^+$14. Suppose that the set of primitive propositions is allowed to be uncountable. Is it true that for every set of propositions there exists an independent set equivalent to it?