Mich. 2024 **LINEAR ANALYSIS – EXAMPLES 1** IBL

1. Prove carefully that $C[0,1]$ is incomplete in the integral norm $\|.\|_1$.

2. Show that $C^1[0,1] = \{f \in C[0,1] : f$ continuously differentiable$\}$ is incomplete in the uniform norm $\|.\|_\infty$ but complete in the norm $\|f\| = \|f\|_\infty + \|f'\|_\infty$.

3. Prove that a normed space X is a Banach space if and only if every series $\sum x_n$ in X with $\sum \|x_n\| < \infty$ is convergent.

4. Let $1 < p, q, r < \infty$ with $1/p + 1/q + 1/r = 1$. Show that if $x \in l_p$, $y \in l_q$, $z \in l_r$ then $\sum |x_n y_n z_n| \leq \|x\|_p \|y\|_q \|z\|_r$.

5. Let $1 < p < \infty$, and let x and y be vectors in l_p with $\|x\| = \|y\| = 1$ and $\|x + y\| = 2$. Prove that $x = y$. Does this result also hold in l_1 or l_∞?

6. Show directly that the spaces l_p, $1 \leq p \leq \infty$, and c_0 are complete.

7. Let x and y be vectors in a normed space X with $\|x\|, \|y\| \geq 1$. Writing x' for $x/\|x\|$ and y' for $y/\|y\|$, is it always true that $\|x' - y'\| \leq \|x - y\|$?

8. Let Y and Z be dense subspaces of a normed space X. Must $Y \cap Z$ be dense in X?

9. Give two inequivalent norms $\|.\|_1$ and $\|.\|_2$ on a vector space V such that the normed spaces $(V, \|.\|_1)$ and $(V, \|.\|_2)$ are isomorphic.

10. Let A and B be subspaces of a vector space V such that $V = A \oplus B$, and let $\|.\|_1$ and $\|.\|_2$ be two norms on V. If $\|.\|_1$ and $\|.\|_2$ are equivalent on A and equivalent on B, must they be equivalent?

11. Let Y be a proper closed subspace of a normed space X. Is there always a non-zero vector $x \in X$ that is 'orthogonal' to Y, in the sense that $\|x + y\| \geq \|y\|$ for all $y \in Y$?

12. Prove that no two of the spaces l_1, l_2, l_∞, c_0 are isomorphic.

13. Does l_∞ contain a subspace isometrically isomorphic to l_2?

$^+$14. Construct two normed spaces X and Y such that $d(X, Y) = 1$ but X and Y are not isometrically isomorphic.