1. Let $1 \le p < q < \infty$. Show that $\ell_p \subsetneq \ell_q \subsetneq \mathrm{c}_0$. Is $\bigcup_{p\in[1,\infty)} \ell_p = \mathrm{c}_0$?

2. Show directly that the spaces ℓ_p, $1 \le p \le \infty$, are complete.

3. Show that the space $C^1[0,1] = \{f \in C[0,1] : f \text{ continuously differentiable}\}$ is incomplete in the uniform norm $\|\cdot\|_\infty$ but complete in the norm $\|f\| = \|f\|_\infty + \|f'\|_\infty$.

4. Show carefully that $\ell_1^* \cong \ell_\infty$ and $\mathrm{c}_0^* \cong \ell_1$.

5. Prove that $C[0,1]$ is separable and that ℓ_∞ is not separable.

6. Prove that a normed space is a Banach space if and only if every series $\sum x_n$ in X with $\sum \|x_n\| < \infty$ is convergent.

7. Let $T\colon \ell_p^n \to \ell_q^n$ be the identity map of the underlying vector space $\mathbb{R}^n$. Compute the operator norm of T for all possible values of p and q.

8. Show that the spaces c_0 and c are isomorphic.

9. Let X be a normed space. For $x \in X \setminus \{0\}$ write $\pi(x) = x/\|x\|$. Is it true that $\|\pi(x) - \pi(y)\| \le \|x - y\|$ whenever $\|x\|,\ \|y\| \ge 1$?

10. Show that no two of the spaces $\ell_1, \ell_2, \ell_\infty, \mathrm{c}_0$ are isomorphic.

11. Let Y and Z be dense subspaces of a normed space X. Is $Y \cap Z$ dense in X?

12. Assume that X is an infinite-dimensional normed space. Show that there is a sequence (x_n) in the unit ball of X with $\|x_m - x_n\| \ge 1$ whenever $m \ne n$. Is it possible to replace $\ge$ by $>$?

13. Does there exist a discontinuous linear map on a Banach space?

$^+$14. Construct two normed spaces X, Y such that $d(X,Y) = 1$ but X and Y are not isometrically isomorphic.