

1. Prove Bessel's inequality: if (e_n) is an orthonormal sequence in an inner product space, then $\sum_n |\langle x, e_n \rangle|^2 \leq \|x\|^2$ for every x .
2. Let H be a Hilbert space. Let Y be a closed subspace of H , and let S be a subset of H . Show that $Y^{\perp\perp} = Y$ and that $S^{\perp\perp} = \overline{\text{span}} S$. Show further that S has dense linear span in H if and only if $S^\perp = \{0\}$.
3. Let $(a_n) \in \ell_\infty$. Define $T: \ell_2 \rightarrow \ell_2$ by $T(x_n) = (a_n x_n)$. Show that $T \in \mathcal{B}(\ell_2)$ and $\|T\| = \|a\|_\infty$. Find the eigenvalues, approximate eigenvalues and the spectrum of T . Show that T is compact if and only if $(a_n) \in c_0$.
4. Show that an operator T on a complex Hilbert space is normal if and only if $\|Tx\| = \|T^*x\|$ for every x .
5. Let U be a unitary operator on a complex Hilbert space. Show that $\sigma(T) \subset \mathbb{T}$.
6. Let H be a complex Hilbert space with orthonormal basis $(e_n)_{n=-\infty}^\infty$. The *bilateral shift* is the operator T on H defined by $Te_n = e_{n+1}$. Find the spectrum of T .
7. Prove that if T is a normal operator then $\sigma(T) = \sigma_{\text{ap}}(T)$.
8. Let S and T be compact normal operators such that $\dim E_S(\lambda) = \dim E_T(\lambda)$ for every λ . Show that there is a unitary operator U such that $T = USU^*$.
9. Are the invertible operators dense in $\mathcal{B}(\ell_2)$?
10. Let X be a complex Banach space and $T \in \mathcal{B}(X)$. Define $r(T)$ for a rational function r with no singularities in $\sigma(T)$. Prove that $\sigma(r(T)) = \{r(\lambda) : \lambda \in \sigma(T)\}$.
11. Let T be an operator on a Banach space X with $\|T\| < 1$. Show that $I - T$ has a square root, i.e., there exists $S \in \mathcal{B}(X)$ such that $S^2 = I - T$.
12. Let $A = (a_{ij})_{i,j=1}^\infty$ be a complex matrix whose rows and columns form a bounded set in ℓ_2 . Must A be the matrix of a bounded linear operator on ℓ_2 ?
13. Construct a linear map T on a (non-zero, complex) vector space V such that $\lambda I - T$ is invertible for every $\lambda \in \mathbb{C}$.
14. Construct a hermitian operator T on a (non-zero) Hilbert space H so that T has no eigenvalues.
15. Let X be a Banach space. Let (x_n) and (f_n) be bounded sequences in X and X^* , respectively, such that $f_i(x_j) = \delta_{ij}$ for all i, j . Let (y_n) be another sequence in X with $\varepsilon = \sum \|x_n - y_n\| < \infty$. Prove that if ε is sufficiently small then there is an invertible operator $T: X \rightarrow X$ such that $Tx_n = y_n$ for all $n \in \mathbb{N}$.
- 16⁺. Let X be a separable infinite-dimensional Banach space. Suppose that there is a constant C such that every finite-dimensional subspace E of X satisfies $d(E, \ell_2^n) \leq C$, where $n = \dim E$. Prove that X is isomorphic to ℓ_2 .