

[Throughout, K is a compact Hausdorff space.]

1. Let $f \in C(K)$. Show that there exists $\phi \in C(K)^*$ with $\|\phi\| = 1$ and $\phi(f) = \|f\|$.
2. A linear map $\phi: C(K) \rightarrow \mathbb{R}$ is said to be *positive* if $\phi(f) \geq 0$ whenever $f \geq 0$. Show that a positive linear map is continuous. What is the norm of ϕ ?
3. Show that $C(K)$ is finite-dimensional if and only if K is finite.
4. Let X be an inner product space, and let $T: X \rightarrow X$ be a linear map. Show that $\langle Tx, Ty \rangle = \langle x, y \rangle$ for all $x, y \in X$ if and only if $\|Tx\| = \|x\|$ for all $x \in X$.
5. Let X be a complex inner product space, and let $T: X \rightarrow X$ be a linear map. Show that if $\langle Tx, x \rangle = 0$ for all $x \in X$, then $T = 0$. Does the same conclusion hold in the real case?
6. Suppose that the family \mathcal{C} of clopen subsets of K is a base for the topology. Show that the indicator functions of members of \mathcal{C} have dense linear span in $C(K)$. [A set is clopen if it is both open and closed.]
7. Let A be a subalgebra of $C^{\mathbb{R}}(K)$ that separates the points of K . Show that either $\overline{A} = C^{\mathbb{R}}(K)$ or there is a point $x_0 \in K$ such that $\overline{A} = \{f \in C^{\mathbb{R}}(K) : f(x_0) = 0\}$.
8. Show that $C(\mathbb{T})$ with the convolution product and the uniform norm is a Banach algebra. Show further that this algebra is non-unital.
9. Let S be a subset of a normal topological space X . Show that there is a continuous function $f: X \rightarrow \mathbb{R}$ such that $S = f^{-1}(0)$ if and only if S is a closed \mathcal{G}_δ set.
10. A topological space is *Lindelöf* if every open cover has a countable subcover, and the space is *regular* if for every closed subset F and for every point $x \notin F$, there are disjoint open sets U and V with $x \in U$ and $F \subset V$. Prove that a regular Lindelöf space is normal.
11. Show that if $C(K)$ is separable, then K is metrizable.
12. Show that there is a sequence (x_n) in the unit ball of ℓ_2 such that $\|x_m - x_n\| > \sqrt{2}$ for all $m \neq n$ in \mathbb{N} . Can the constant $\sqrt{2}$ be improved?
13. Construct an inner product space X and a proper closed subspace Y of X such that $Y^\perp = \{0\}$.
14. A series $\sum x_i$ in a Banach space converges *unconditionally* if $\sum \varepsilon_i x_i$ converges for all choices of signs $\varepsilon_i = \pm 1$. Show that if this happens in a Hilbert space then $\sum \|x_i\|^2 < \infty$.
15. Let H be a Hilbert space, and let $f: [0, 1] \rightarrow H$ be a continuous function. Suppose that for every $x < y < z$ we have $f(x) - f(y) \perp f(y) - f(z)$. Must f be constant?
16. Is there a continuous surjective map $f: \mathbb{R} \rightarrow \ell_2$?