

1. Let $p < \infty$ and $x \in \ell_p$. Show that there is an $f \in \ell_p^*$ with $\|f\| = 1$ and $f(x) = \|x\|$.
2. Let f be a linear functional on a normed space X . Prove that f is continuous if and only if $\ker f$ is closed.
3. Let $A \subset Y \subset X$ with A nowhere dense in Y . Show that A is nowhere dense in X .
4. Prove Osgood's theorem: if (f_n) is a sequence of continuous functions $[0, 1] \rightarrow \mathbb{R}$ such that $(f_n(t))$ is bounded for every $t \in [0, 1]$, then there is an interval $[a, b]$ with $a < b$ on which the f_n are uniformly bounded.
5. Let X be a closed subspace of ℓ_1 . Assume that every $y = (x_{2n}) \in \ell_1$ extends to a sequence $x = (x_n) \in X$. Show that there is a constant C such that x can always be chosen to satisfy $\|x\| \leq C\|y\|$.
6. Assume that X is a closed subspace of $C[0, 1]$ such that every element of X is continuously differentiable. Show that X is finite-dimensional.
7. Suppose that $T: X \rightarrow Y$ satisfies the conditions in the Open Mapping Lemma. Show that Y is complete.
8. Let Y be a proper subspace of a Banach space X . Can Y be dense \mathcal{G}_δ , *i.e.*, can Y be the intersection of a sequence of dense open sets in X ?
9. Deduce the Baire Category Theorem from the fact that a non-empty complete metric space is of second category in itself. (This is more subtle than it looks!)
10. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function such that for every $x > 0$ we have $f(nx) \rightarrow 0$ as $n \rightarrow \infty$. Show that $f(x) \rightarrow 0$ as $x \rightarrow \infty$.
11. Give a direct, elementary proof of the Principle of Uniform Boundedness (*i.e.*, one that does not use the Baire Category Theorem).
12. Does there exist a function $f: [0, 1] \rightarrow \mathbb{R}$ which is continuous at every rational and discontinuous at every irrational?
13. Let $f: [0, 1] \rightarrow \mathbb{R}$ be a pointwise limit of a sequence of continuous functions. Show that f has a point of continuity.
14. Assume that X is a closed subspace of $L_2[0, 1]$ such that every element of X is also in $L_\infty[0, 1]$. Show that X is finite-dimensional.
- +15. Let X be a normed space that is homeomorphic to a complete metric space. Prove that X is complete.
- +16. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be an infinitely differentiable function such that for every $x \in \mathbb{R}$ there is an $n \in \mathbb{N}$ with $f^{(m)}(x) = 0$ for all $m \geq n$. Prove that f is a polynomial.