1. Define *normed vector space*, *Banach space*, *inner product (Euclidean) space*, *Hilbert space*. Give examples of a normed vector space which is not a Banach space, a Banach space which does not arise from an inner product (Euclidean) space, and an inner product (Euclidean) space which is not a Hilbert space. What is the relation of these concepts in the finite dimensional case?

State the *inverse mapping theorem* for maps $T : \mathcal{X} \to \mathcal{Y}$ where $\mathcal{X}$, $\mathcal{Y}$ are Banach spaces. Provide a counterexample in the case where $\mathcal{X}$ and $\mathcal{Y}$ are assumed only to be normed vector spaces.

2. State (without proof) the *Stone-Weierstrass Theorem*. Describe the closure in $C[0, 2\pi]$ of the sets listed below:

1. The set of trigonometric polynomials, i.e. the set of polynomials in $\sin\theta$, $\cos\theta$.
2. The set of all polynomials vanishing at 0.
3. The set of all polynomials $\{p : p(0) = 0 \text{ or } p(1) = 0\}$.
4. The set of all polynomials with rational coefficients.
5. The set of piecewise linear functions.

Justify your answers.

3. Let X be a topological space, and let f_i be a collection of functions $f_i : X \to \mathbb{R}$. Define what it means for $\{f_i\}$ to be *uniformly bounded* on X. Prove Osgood's theorem: Let f_1, f_2,... be continuous functions $[0, 1] \mapsto \mathbb{R}$ such that for each $x \in [0, 1]$, the set $\{f_i(x)\}$ is bounded. Then there is a non-empty subinterval $[a, b]$, with $a < b$, such that $\{f_i\}$ is uniformly bounded. Show by explicit counterexample that the assumption of continuity cannot be dropped.

4. Let $\mathcal{X}$ be a metric space. Define what it means for a subset $Y \subset \mathcal{X}$ to be of *first category* and of *second category*. If $\mathcal{X}$ is complete, and Y is closed and non-empty, of what category is it? Let the subset $Y \subset [0, 1]$ be defined by

$$Y = \{x \in [0, 1] \setminus \mathbb{Q} : \forall_{n\geq 0}\exists_{0\leq p\leq q, q>n} : |x - p/q| < 1/q^3\}.$$

In the above p and q are integers. Is $Y = \emptyset$? Is $Y = [0, 1] \setminus \mathcal{Q}$? Justify your answers.

5. Let V be a finite dimensional complex vector space, and let T be a linear operator $T : V \to V$. Choose a norm on V. This makes V into a Banach space. (Why?) State the definition of the *spectrum* of T, denoted $\sigma(T)$. What can be said about $\sigma(T)$, in view of finite dimensionality? Does the spectrum depend on the choice of norm above?

Alternatively, choose an inner product $e(\cdot, \cdot)$ on V. This makes V into a Hilbert space. (Why?) State what it means for T to be *self-adjoint* with respect to e. Suppose now T is indeed self-adjoint. Prove the spectral theorem for T, i.e. prove that there exist orthogonal unit vectors $v_1, \ldots, v_n$, where n is the dimension of V, and real numbers $\lambda_1, \ldots, \lambda_n$, such that

$$T(v) = \sum_i \lambda_i e(v_i, v) v_i,$$

where $\{\lambda_i\} = \sigma(T)$. Are v_i necessarily unique, up to reordering?

For comments, email `M.Dafermos@dpmms.cam.ac.uk`