Graph Theory (2013–14) Example Sheet 5 of 4

Extra examples for the interested: not for supervisions.

1. Let G be a regular graph of order n. Show that the largest complete subgraph of G has order $1, 2, \ldots, \lfloor n/2 \rfloor$ or n, and that each of these is possible.

2. Let G be a graph on vertex set V. Show that there is a partition $V_1 \cup V_2$ of G such that in each of $G[V_1]$ and $G[V_2]$ all vertices have even degree.

3. Let t_n be the number of labelled trees on vertex set $[n]$. So there are $n^2 t_n$ objects (u, v, T) where $u, v \in [n]$ and T is a tree. Let $P = x_1 x_2 \ldots x_k$ be the u–v path in T, where $u = w_1$, $v = w_k$ (we allow $u = v$). Define $f : V(P) \to V(P)$ so that $f(x_1) < f(x_2) < \cdots < f(x_k)$ (remember $V(P) \subset [n]$). Extend f to $f : [n] \to [n]$ by defining $f(y) = z$ where z is the neighbour of y closest to P. Hence prove $n^2 t_n = n^n$, that is, $t_n = n^{n-2}$.

4. Show that there are n^{n-3} trees with n unlabelled vertices and $n-1$ labelled edges.

5. In a connected graph G, let $\pi(G)$ be the minimum number of edges meeting every vertex and $\beta(G)$ be the maximum sumber of independent edges. Prove that $\pi(G) + \beta(G) = |G|$.

6. Draw the maps of the five Platonic solids. What are the dual maps?

7. Let G be a connected, bridgeless plane graph drawn with straight edges. Reprove Euler's formula by evaluating the sum of all angles in all faces of G in two different ways. What can you say if G is drawn on the torus instead of the plane?

8. Show that if $\kappa(G) \geq 3$ then G contains a subdivision of K_4. Deduce that if $e(G) \geq 2|G| - 2$ then G contains a subdivision of K_4.

9. Let $\theta(G)$ be the minimum number of planar graphs into which G can be decomposed. Show $\theta(K_{4n-1,4n-1}) \geq n + 1$. Find $\theta(K_{7,7})$.

10. Let H be a subgroup of the finite group G having index k. Show that there are elements $g_1, \ldots, g_k \in G$ such that $g_1 H, \ldots, g_k H$ are the distinct left cosets and $H g_1, \ldots, H g_k$ are the distinct right cosets. (P. Hall)

11. Let $A = (a_{ij})_1^n$ be an $n \times n$ doubly stochastic matrix, that is, its entries are non-negative and the rows and columns each sum to one. Show that A is in the convex hull of the set of $n \times n$ permutation matrices, i.e. there are permutation matrices $P_1, P_2, \ldots, P_m$ such that $A = \sum_1^m \lambda_i P_i$ and $\sum_1^m \lambda_i = 1$.

12. Show that Hall's condition will not ensure a matching in an infinite bipartite graph, but that it will if G is countable and every $x \in X$ has finite degree. $^+$What if G is uncountable?

$^+$**13.** Prove Petersen's theorem: every cubic multigraph with at most one isthmus has a 1-factor.

14. The *independence number* $\beta(G)$ of a graph is the size of a largest independent vertex subset (spanning no edges). Show that if $\beta(G) \leq \kappa(G)$ then G is Hamiltonian.

15. How many edges can a graph of order n containing precisely one triangle have?

16. The Tutte 8-cage has vertex set $\{n_1, n_3, n_5 : n \in [10]\}$, with edges n_1n_5, n_3n_5 and n_i to m_i iff $n - m = \pm i \pmod{10}$. Show that any path of length 5 is equivalent to any other under some automorphism. Find the eigenvalues of the Tutte 8-cage.

17. Let G be a graph in which every edge is in a unique triangle and every non-edge is a diagonal of a unique 4-cycle. Show that $|G| \in \{3, 9, 99, 243, 6273, 494019\}$.

18. Show $\chi(G) \leq 4$ if all odd cycles are triangles.

19. Let (G_n) be a sequence of graphs with $|G_n| = n$ for each n. If there is some $\epsilon > 0$ such that we have $e(G_n) > (\frac{2}{3} + \epsilon)\binom{n}{2}$ for every n, why must every planar graph be a subgraph of some G_n? Show that this need not be the case if instead $e(G_n)/\binom{n}{2} \to 2/3$.

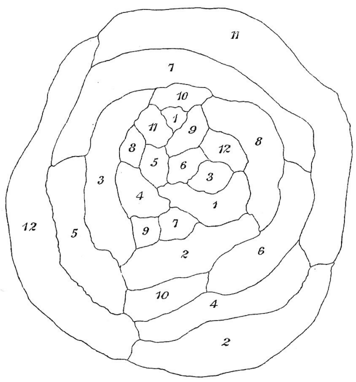
20. Show that $|p_G(-1)| = (-1)^{|G|}p_G(-1)$ is the number of acyclic orientations of G.

(An orientation of G is an assignment of a direction to each edge.)

21. Let G be the graph obtained by subdividing a single edge of $K_{n,n}$ by a new vertex. Show that $\chi'(G) = \Delta(G) + 1$, but that if e is any edge of G then $\chi'(G - e) = \Delta(G - e)$.

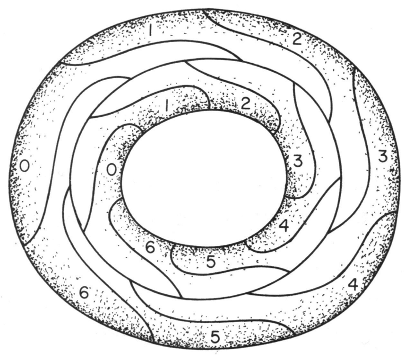
22. Show that a bipartite graph G contains an independent set of edges meeting every vertex of maximum degree. Deduce that $\chi'(G) = \Delta(G)$.

23. In the colouring of a plane map, an *m-pire* is a set of up to m faces that must receive the same colour (eg France and its overseas departments). Show that if a map has m-pires it can be coloured with $6m$ colours. Find a 2-pire map that needs 12 colours.



left:
Heawood's 2-pire map

right:
Heawood's hoop



24. Show that a planar cubic graph is face 3-colourable if and only if each face has even length.

25. Show that $R_3(3,3,3) \leq 17$. Give an example to show that $R_3(3,3,3) = 17$.

26. Show that there is an infinite set S of positive integers such that the sum of any two distinct elements of S has an even number of distinct prime factors.

27. By choosing a subset of $V(G)$ randomly with probability $p = \log(\delta+1)/(\delta+1)$, where $\delta = \delta(G)$, show that the graph G has a subset $U \subset V(G)$ such that every vertex in $V(G) - U$ has a neighbour in U and $|U| \leq pn + n(1-p)^{\delta+1} \leq n(1 + \log(\delta+1))/(\delta+1)$.

+**28.** The group of all isomorphisms from a graph G to iself is called the *automorphism group* of G. Show that every finite group is the automorphism group of some graph. Is every group the automorphism group of some (possibly infinite) graph?