Mich. 2010 **GRAPH THEORY—EXAMPLES 4** PAR

1. Let G be a graph and $v \in G$. Must $\kappa(G-v) \leqslant \kappa(G)$?

2. Show that, for any graph G, $\kappa(G) \leqslant \lambda(G)$, and that if G is 3-regular then $\kappa(G) = \lambda(G)$. Given positive integers $k \leqslant \ell$, construct a graph G with $\kappa(G) = k$ and $\lambda(G) = \ell$.

3. For a set $B \subset V(G)$ and a vertex a not in B, an *a-B fan* is a family of $|B|$ paths from a to B, any two meeting only at a. Show that a graph G (with $|G| > k$) is k-connected if and only if there is an a-B fan for every $B \subset V(G)$ with $|B| = k$ and every vertex a not in B.

4. Let G be a k-connected graph ($k \geqslant 2$), and let $x_1, x_2, \ldots, x_k$ be vertices of G. Show that there is a cycle in G containing all the x_i.

5. Show directly (without using expectation and variance methods) that if $p \in (0,1)$ is constant then a. e. $G \in \mathcal{G}(n,p)$ contains a triangle, whereas if $p = n^{-2}$ then a. e. $G \in \mathcal{G}(n,p)$ is triangle-free.

6. Prove that, for any k, there is a tournament in which, for any k players, there is a player who beats all of them. Exhibit such a tournament for $k = 2$.

7. Let G be a (not necessarily planar) graph with $|G| = n$ and $e(G) = m$. Suppose that G is drawn in the plane, but with edges allowed to cross. Let t be the number of pairs of edges which cross. Show that $t \geqslant m - 3n + 6$

Suppose now $m \geqslant 4n$. By considering a random set $W \subset V(G)$ containing each vertex of G independently with probability $4n/m$, show that in fact $t \geqslant m^3/64n^2$.

8. Let $p = \lambda \log n/n$ where $\lambda > 0$ is constant. Show that if $\lambda < 1$ then a. e. $G \in \mathcal{G}(n,p)$ has an *isolated vertex,* i. e. a vertex of degree zero, whereas if $\lambda > 1$ then a. e. $G \in \mathcal{G}(n,p)$ has no isolated vertex.

9 Show that $R(s,t) \geqslant n - \binom{n}{s} p^{\binom{s}{2}} - \binom{n}{t}(1-p)^{\binom{t}{2}}$ for all $n \in \mathbb{N}$ and $p \in (0,1)$. By choosing p appropriately, deduce that $R(4,t) = \Omega\big((t/\log t)^{\frac{3}{2}}\big)$.

10. Find the eigenvalues of the complete r-partite graph $K_r(t)$ and of the cycle C_n.

11.Let G be a graph in which every edge is in a unique triangle and every non-edge is a diagonal of a unique 4-cycle. Show that $|G| \in \{3, 9, 99, 243, 6273, 494019\}$.

$^+$12. Any two members of a certain College have a unique common enemy who is also a member of the College. Show that there is some College member (the 'Junior Bursar') who is everyone else's enemy.