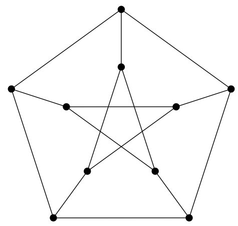
Mich. 2010 **GRAPH THEORY—EXAMPLES 2** PAR



1. Show that a connected graph $G$ with at least one vertex must contain a vertex $v$ such that $G - v$ is connected.

2. For which $m$ and $n$ is the complete bipartite graph $K_{m,n}$ Hamiltonian?

3. What is $\lim_{n\to\infty} ex(P; n)/\binom{n}{2}$, where $P$ is the Petersen graph (shown)?

4. For each $r \geq 3$, construct a graph of chromatic number $r$ which contains no $K_r$.

5. Without assuming Turán's theorem, show that the Turán graph $T_r(n)$ is the unique $r$-partite graph of order $n$ with the largest possible number of edges.

6. Let $G$ be a graph. Define a relation $\to$ on $V(G)$ by $u \to v$ iff there is a path in $G$ from $u$ to $v$. Show that $\to$ is an equivalence relation. Hence deduce that $G$ can be written uniquely as a disjoint union of connected subgraphs each with at least one vertex.

7. Let $G$ be a graph of order $n$. Suppose that $d(x) + d(y) \geq n$ for all distinct, non-adjacent $x$, $y \in G$. Show that $G$ is Hamiltonian. Deduce that any graph of order $n \geq 3$ with $\binom{n}{2} - n + 3$ edges must be Hamiltonian. For each $n \geq 3$, give an example of a graph with $\binom{n}{2} - n + 2$ edges which is not Hamiltonian.

8. What is $ex(K_{1,s}; n)$?

9. Let $x_1$, $x_2$, $\ldots$, $x_{3n}$ be points in the plane such that no two of them are more than distance 1 apart. Prove that at most $3n^2$ of the distances $\|x_i - x_j\|$ $(i < j)$ are greater than $1/\sqrt{2}$.

10. A *deleted* $K_r$ consists of a $K_r$ from which an edge has been removed. Show that if $G$ is a graph of order $n \geq r + 1$ with $e(G) > t_{r-1}(n)$ then $G$ contains a deleted $K_{r+1}$.

11. For each positive integer $n$, let $g_n$ be the largest integer $k$ such that it is possible to colour $k$ edges of the complete graph $K_n$ blue or yellow without creating a monochromatic (blue or yellow) triangle. Show that $g_n/\binom{n}{2}$ converges, and find $\lim_{n\to\infty} g_n/\binom{n}{2}$.

12. How many edges can a graph $G$ of order $n$ containing precisely one triangle have?

13. The *upper density* $\mathrm{ud}\,(G)$ of an infinite graph $G$ is the supremum of the densities of its large finite subgraphs; that is to say,

$$\mathrm{ud}\,(G) = \lim_{n\to\infty} \left( \sup \left\{ e(H) \Big/ \binom{n}{2} : H \subset G, |H| = n \right\} \right).$$

Show that the upper density of every infinite graph (is well-defined and) lies in the set $\{0, 1, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots\}$.

$^+$14. Show that an $r$-regular graph of order $2r + 1$ must be Hamiltonian.

$^+$15. Construct a triangle-free graph of chromatic number 1526.