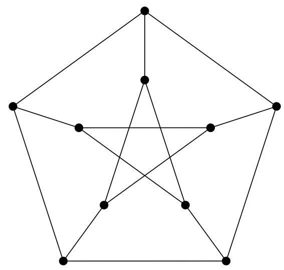
Mich. 2009 **GRAPH THEORY—EXAMPLES 2** PAR



1. How many spanning trees does K_4 have?

2. Show that a connected graph G with at least one vertex must contain a vertex v such that $G - v$ is connected.

3. For which m and n is the complete bipartite graph $K_{m,n}$ Hamiltonian? Is the Petersen graph (shown) Hamiltonian?

4. Show that any graph of order $n \geq 3$ with $\binom{n}{2} - n + 3$ edges must be Hamiltonian. For each $n \geq 3$, give an example of a graph with $\binom{n}{2} - n + 2$ edges which is not Hamiltonian.

5. What is $\lim_{n\to\infty} ex(P;n)/\binom{n}{2}$, where P is the Petersen graph?

6. For each $r \geq 3$, construct a graph of chromatic number r which contains no K_r.

7. Let x_1, x_2, ..., x_{3n} be points in the plane such that no two of them are more than distance 1 apart. Prove that at most $3n^2$ of the distances $\|x_i - x_j\|$ $(i < j)$ are greater than $1/\sqrt{2}$.

8. A *deleted* K_r consists of a K_r from which an edge has been removed. Show that if G is a graph of order n $(n \geq r+1)$ with $e(G) > e(T_{r-1}(n))$ then G contains a deleted K_{r+1}.

9. For each positive integer n, let g_n be the largest integer k such that it is possible to colour k edges of the complete graph K_n red or blue without creating a monochromatic (red or blue) triangle. Show that $g_n/\binom{n}{2}$ converges, and find $\lim_{n\to\infty} g_n/\binom{n}{2}$.

10. How many edges can a graph G of order n containing precisely one triangle have?

11. The *upper density* $\mathrm{ud}\,(G)$ of an infinite graph G is the supremum of the densities of its large finite subgraphs; that is to say,

$$\mathrm{ud}\,(G) = \lim_{n\to\infty} \left(\sup \left\{ e(H) \middle/ \binom{n}{2} : H \subset G, |H| = n \right\} \right).$$

Show that the upper density of every infinite graph (is well-defined and) lies in the set $\{0, 1, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots\}$.

12. Show that an r-regular graph of order $2r+1$ must be Hamiltonian.

$^{+}$13. Construct a triangle-free graph of chromatic number 1526.