MATHEMATICAL TRIPOS PART II (2004–05)

Graph Theory - Problem Sheet 4 of 4 **A.G. Thomason**

Note: The exercises are largely independent of each other — so, if you can't do one, go on to another.

48) Prove that $R(3,3) = 6$ and $R(3,4) = 9$. For each $t \geq 2$ construct a $t-$regular graph which shows that $R(3, t+1) \geq 3t$.

49) By considering the graph on $\mathbb{F}_{17}$ (the field of integers modulo 17) in which x is joined to y if $x - y$ is a square (modulo 17), show that $R(4,4) = 18$.

50) Show that $R_3(3,3,3) \leq 17$. Give an example to show that $R_3(3,3,3) = 17$.
[Try it by hand. But this works: take as vertices the field $\mathbb{F}_{2^4}$, let g be a primitive root, and colour uv with i (modulo 3) where $u - v = g^i$.]

51) Show that $R_k(3,3,\ldots 3) \leq \lfloor ek! \rfloor + 1$.
Deduce the following theorem of Schur: if we partition the numbers $1, 2, \ldots, \lfloor ek! \rfloor$ into k classes then the equation $x + y = z$ is soluble in at least one of the classes.
Conclude that, for any fixed n, the "Fermat" equation $x^n + y^n \equiv z^n \pmod p$ has a non-trivial solution (that is, $xyz \not\equiv 0$) for all sufficiently large primes p.

52) Let A be a set of $R^{(4)}(n,5)$ points in the plane, with no three points of A collinear. Prove that A contains n points forming a convex n-gon.
Give a different argument to prove the same with $R^{(3)}(n,n)$ in place of $R^{(4)}(n,5)$.
Show that every infinite set of points in the plane, with no three on a line, contains an 'infinite convex polygon' S, meaning no point of S is in the convex hull of the others.

53) Given a finite graph G, let $R(G)$ be the smallest n such that every red-blue colouring of K_n yields a monochromatic copy of G. [Note: $R(G)$ exists since $R(G) \leq R(|G|)$.]
 (i) Let I_k be a set of k independent edges, so $|I_k| = 2k$. Show $R(I_k) = 3k - 1$.
 (ii) Let H_k consist of a triangle xyz and k edges $xx_1, xx_2, \ldots, xx_k$, so $|H_k| = k + 3$. Show that $R(H_1) = 7$. What is $R(H_k)$?
 (iii) Show that $R(C_4) = 6$.

54) Prove that every sequence of $mn + 1$ numbers contains an increasing subsequence of length $m + 1$ or a decreasing sequence of length $n + 1$. [Imagine putting the numbers in piles, placing a number on top of the first pile whose current top is smaller.]

55) By painting its vertices red or blue at random, show that a graph G has a bipartition $V(G) = V_1 \cup V_2$ such that $e(G[V_1]) + e(G[V_2]) \leq \frac{1}{2}e(G)$ (c.f. example sheet 1).

56) Prove that there is a tournament (see example sheet 1) of order n containing at least $2^{-n}(n-1)!$ directed hamiltonian cycles.

57) The *crossing number* $\xi(G)$ of a graph G is the minimum number of edge crossings in a drawing of G in the plane. Let $|G| = n$ and $e(G) = m$. Show that $\xi(G) \geq m - 3n + 6$.
Improve this when $m \geq 4n$ as follows. Choose a random subset $S \subset V(G)$ by choosing vertices independently with probability $p = 4n/m$. Let $X_S = \xi(G[S]) - e(G[S]) + 3|S|$, so $X_S \geq 0$. Show $\mathrm{E}X_S = p^4\xi(G) - p^2 m + 3pn$. Thus $\xi(G) \geq m^3/64n^2$.

58) By choosing a subset of $V(G)$ randomly with probability $p = \log(\delta+1)/(\delta+1)$, where $\delta = \delta(G)$, show that the graph G has a subset $U \subset V(G)$ such that every vertex in $V(G) - U$ has a neighbour in U and $|U| \leq pn + n(1-p)^{\delta+1} \leq n(1 + \log(\delta+1))/(\delta+1)$.

59) Use Stirling's formula to show that $\binom{n}{s}2^{1-\binom{s}{2}} < 1$ if $n = ((1-\epsilon)/\sqrt{2}e)s2^{s/2}$ and s is large. Conclude that $R(s) > (1/\sqrt{2}e + o(1))s2^{s/2}$.

By removing a vertex from each monochromatic K_s in a random colouring, show that $R(s) > n - \binom{n}{s}2^{1-\binom{s}{2}}$ for every n. Conclude that $R(s) > (1/e + o(1))s2^{s/2}$.

60) Show that for every $n \geq 1$ there is an $n \times n$ bipartite graph of size at least $\frac{1}{2}n^{2-\sigma}$ which contains no $K_{s,t}$, where $\sigma = (s+t-2)/(st-1)$.

61) Let X be the number of vertices of degree 1 in $G \in \mathcal{G}(n,p)$. Show that $\mathrm{E}X = n(n-1)p(1-p)^{n-2}$. Hence show that, if $\omega(n) \to \infty$ and $p = (\log n + \log\log n + \omega(n))/n$, then $X = 0$ almost surely. Show that $\mathrm{Var}X/\mathrm{E}X = 1 + (1-p)^{n-2} + (n-2)^2p(1-p)^{n-3} - \mathrm{E}X$, and hence that if $p = \log n/n$ then $X \neq 0$ almost surely.

62) Let X be the number of K_4's in $G \in \mathcal{G}(n,p)$. Show that $\mathrm{E}X = \binom{n}{4}p^6$ and that $\mathrm{Var}X/\mathrm{E}X = (1-p^6) + 4(n-4)(p^3-p^6) + 6\binom{n-4}{2}(p^5-p^6)$. Hence show that $p/n^{-2/3} \to 0$ then $X = 0$ almost surely, whereas if $p/n^{-2/3} \to \infty$ then $X \neq 0$ almost surely.

63) Show that, if $G = K_n$, then A has eigenvalues $n-1$ (once) and -1 ($n-1$ times). Show that if $G = K_{s,t}$ then A has eigenvalues $\pm\sqrt{st}$ (once each) and 0 ($s+t-2$ times), and L has eigenvalues 0 and $s+t$ (once each), s ($t-1$ times) and t ($s-1$ times).

64) Let B be an incidence matrix for G. Show that G has $|G| - \mathrm{rank}(B)$ components.

65) Let G be a graph in which every edge is in a unique K_3 and every non-edge is the diagonal of a unique C_4. Show that $|G| = 1 + 2t^2$ and that G is strongly regular with parameters $(2t, 1, 2)$, for some $t \in \{1, 2, 7, 11, 56, 497\}$.

$^+$66) You are at a party where you know at least as many people as anyone else does. You discover that every two people there have exactly one mutual acquaintance at the party. Prove that you know everybody else.

Further Problems

Note: the examples above are minimal to cover the course; you are encouraged to do those below also.

F19) Let the infinite subsets of $\mathbb{N}$ be 2-coloured. Must there exist an infinite set $M \subset \mathbb{N}$ all of whose infinite subsets have the same colour?

$^+$F20) Let A be an uncountable set, and let $A^{(2)}$ be 2-coloured. Must there exist an uncountable monochromatic set in A?

F21) Show that $R(s,t) > n - \binom{n}{s}p^{\binom{s}{2}} - \binom{n}{t}(1-p)^{\binom{t}{2}}$ for every n and p. By taking $p = n^{-2/3}$, deduce that $R(4,t) > (t/3\log t)^{3/2}$ for large t.

F22) Let $\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \ldots + a_n$ be the characteristic polynomial of the adjacency matrix of G. Show that $a_1 = 0$, $a_2 = -e(G)$ and $a_3 = -2 \times$ the number of K_3's in G.

F23) Let G be a graph of order n and size m, and B the incidence matrix of some orientation. Let $\widetilde{B}$ be the $(n-1) \times m$ matrix obtained by deleting some row of B. For each set S of $n-1$ edges of G let P_S be the corresponding $(n-1) \times (n-1)$ submatrix of $\widetilde{B}$.

(i) Show that $\det P_S = \pm 1$ if S forms a spanning tree and $\det P_S = 0$ otherwise.

(ii) The Cauchy-Binet formula states that $\det \widetilde{B}\widetilde{B}^t = \sum_S \det P_S P_S^t$, where S runs over all subsets of $n-1$ edges. Deduce that G has $n^{-2}\det(L+J)$ spanning trees.

(iii) By taking $G = K_n$, show there are n^{n-2} labelled trees of order n.