**Geometry & Groups, 2014 – Sheet 4**

1. Construct a Schottky group (i.e. a Kleinian group generated by Möbius maps which pair suitable disjoint discs) whose limit set is contained in the real line.

2. Let $G \leq$ Möb be a Schottky group generated by maps pairing discs with disjoint closures.

   (i) Prove that $G$ contains no elliptic or parabolic elements.

   (ii) Prove that the limit set $\Lambda(G)$ is totally disconnected.

   (iii) Explain why the quotient $\mathbb{H}^3/G$ is a "handlebody" (the open region in space bound by a surface of some genus $\geq 1$).

3. (i) Show that the maps $x \mapsto x/3$ and $x \mapsto 2/3 + x/3$ have non-empty invariant sets other than the middle-thirds Cantor set.

   (ii) Find two similarities $S_1$, $S_2$ of $\mathbb{R}$ such that the unit interval $[0,1]$ is the unique non-empty compact invariant set for the $S_i$.

   (iii) Write the Cantor set $C$ as the invariant set of a collection of *three* similarities of $\mathbb{R}$, and hence (re-)compute its Hausdorff dimension.

4. Let $\mathbb{Z}_2^{\mathbb{N}}$ denote the space of sequences $\mathbf{x} = (\mathbf{x_0}, \mathbf{x_1}, \mathbf{x_2}, \ldots)$ with $x_j \in \{0,1\}$ for every $j$. Define a metric on $\mathbb{Z}_2^{\mathbb{N}}$ by

$$d(\mathbf{x},\mathbf{y}) = \mathbf{2^{-n}} \qquad \text{when } \mathbf{n} = \min\{\mathbf{k} \,|\, \mathbf{x_k} \neq \mathbf{y_k}\}$$

   (and $d(\mathbf{x},\mathbf{y}) = \mathbf{0}$ if $\mathbf{x} = \mathbf{y}$). Construct a homeomorphism from $(\mathbb{Z}_2^{\mathbb{N}}, d)$ to the Cantor middle-thirds set $C$. Describe the self-similarities of $C$ in terms of this space of sequences.

5. (i) Let $F$ be a finite subset of $\mathbb{R}^n$. Show that the zero-dimensional Hausdorff measure $\mathcal{H}^0(F)$ is the cardinality of $F$.

   (ii) Show that for infinitely many (or even every) $s \in [0,2]$ there is a totally disconnected subset $F \subset \mathbb{R}^2$ for which $\dim_H(F) = s$.

6. (i) Compute the Hausdorff dimension of the Sierpinski carpet, given by cutting a square into nine equal pieces, and removing the central one.

   (ii) Let $F = \{x \in \mathbb{R} \,|\, x = b_m b_{m-1} \ldots b_1 . a_1 a_2 \ldots \text{ with } b_i, a_j \neq 5\}$ be those points on the line which admit decimal expansions omitting the number 5. What is $\dim_H(F)$?

   (iii) Construct a fractal in the plane whose Hausdorff dimension is given by the positive real solution $s$ to the equation $4(\frac{1}{4})^s + (\frac{1}{2})^s = 1$.

7. (i) Suppose $F \subset \mathbb{R}^n$ is written as $F = \cup_{i \in \mathbb{Z}} F_i$. Show $\dim_H(F) = sup_i\{\dim_H(F_i)\}$.

(ii) Deduce that if $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable, then for every subset $F \subset \mathbb{R}$, $\dim_H(f(F)) \leq \dim_H(F)$.

(iii) Now consider $f : \mathbb{R} \to \mathbb{R}$ taking $x \mapsto x^2$. By considering the square-root function $\mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, or otherwise, show that for every $F \subset \mathbb{R}$, $\dim_H(f(F)) = \dim_H(F)$.

8. Let $(X, d)$ be a metric space and $\cdots \supset A_n \supset A_{n+1} \supset \cdots$ be a sequence of decreasing non-empty compact subsets of $X$. Prove that the intersection $\cap_k A_k$ is non-empty and compact. If the $A_k$ were non-empty and open would the intersection necessarily be (i) non-empty (ii) open ?

9. Show that Hausdorff distance $d_{Haus}(A, B)$ defines a metric space structure on the set of compact subsets of a given metric space.

   [Recall $d_{Haus}(A, B) = \inf\{\delta \,|\, A \subset \mathcal{U}_\delta(B), B \subset \mathcal{U}_\delta(A)\}$, where $\mathcal{U}_\delta$ denotes the metric $\delta$-neighbourhood.]

10. Give explicit examples of Kleinian groups realising 3 different values of Hausdorff dimension for their limit sets. Justify your answer!

The final two questions are optional extras.

A (Fractals ubiquitous)

(i) Let $S_i : \mathbb{R}^n \to \mathbb{R}^n$ be contractions of fixed factor $c \in (0, 1)$ for $1 \leq i \leq m$. Let $E \subset \mathbb{R}^n$ be any non-empty compact set and let $F$ be the invariant set for the $\{S_j\}$. Show that for the Hausdorff distance:

$$d_{Haus}(E, F) \leq \frac{1}{1-c} d_{Haus}(E, \cup_{j=1}^m S_j(E))$$

(ii) Fix any non-empty compact set $E \subset \mathbb{R}^n$ and $\delta > 0$. Considering a covering of $E$ by a finite set of balls, find contracting similarities $\{S_i, 1 \leq i \leq m\}$ for which $E \subset \bigcup_j \mathcal{U}_{\delta/2}(S_j(E))$ and $\bigcup_j S_j(E) \subset \mathcal{U}_{\delta/2}(E)$. Deduce that $d_{Haus}(E, F) < \delta$ for the "fractal" invariant set $F$ of the $\{S_j\}$. Upshot: every $E$ can be approximated by fractals.

B (Weighing dust)

Take a classical Schottky group $G$ (on disjoint circles $C_i$), with limit set a Cantor dust $\Lambda$. For each real $r > 0$ let $N(r)$ be the (finite!) number of image circles $\{g(C_i) \,|\, g \in G\}$ with Euclidean radius $> r$. Explain *heuristically* why $N(r) \approx (const/r)^s$ for $s = \dim_H(\Lambda)$ and $r \ll 1$. Hence we expect $\dim_H(\Lambda) = \lim_{r \to 0}(-\log N(r)/\log r)$.

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