Geometry & Groups, 2014 – Sheet 3

1. Let J denote inversion in the unit sphere $S^2 \subset \mathbb{R}^3$. If Σ is any sphere in $\mathbb{R}^3$, show that Σ is orthogonal to S^2 if and only if the inversions in S^2 and Σ commute, i.e. $J \circ J_\Sigma = J_\Sigma \circ J$.

2. Let f, g be Möbius maps. Suppose g is loxodromic, and that f and g have exactly one common fixed point. Prove that the subgroup $\langle f, g \rangle$ generated by f and g is not discrete.

3. Suppose $G \leq \text{Möb}(\mathbb{D})$ is discrete and acts properly discontinuously in $\mathbb{D}$. How are the fundamental domains for G acting on $\mathbb{D}$ and for G acting on hyperbolic 3-space $\mathbb{H}^3$ related?

4. Let l, l' be geodesics in $\mathbb{H}^n$, $n = 2, 3$.

   (i) Draw the set of points which are a bounded distance δ away from l.

   (ii) Draw the set of points which are equidistant from l, l'.

   (iii) Suppose now l, l', l'' are geodesics in $\mathbb{H}^2$ which meet pairwise, bounding a hyperbolic triangle. Prove that the angle bisectors of the triangle meet at a point.

   (iv) If 3 geodesics $l, l', l'' \subset \mathbb{H}^3$ meet pairwise, must they lie in a copy of the hyperbolic plane $\mathbb{H}^2 \subset \mathbb{H}^3$?

5. The *translation length* $\eta(g)$ of an orientation-preserving isometry $g \in \text{Isom}^+(\mathbb{H}^3)$ is $\inf_{p \in \mathbb{H}^3} d_{hyp}(p, g(p))$.

   (i) Show that this is achieved along the axis of the isometry, if there is an axis. What happens if there is no axis?

   (ii) Show the translation length of $m_k : z \mapsto kz$ is $\log |k|$.

   (iii) Find the translation length of $z \mapsto \frac{2z+1}{5z+3}$.

6. Two distinct geodesics l_1, l_2 in $\mathbb{H}^3$ are *parallel* if they have a common end-point on $\partial\mathbb{H}^3 = S^2$.

   (i) Show that if l_1 and l_2 are not parallel, the minimum distance

   $$\inf\{d_{hyp}(x, y) \mid x \in l_1, y \in l_2\}$$

   is realised. Are the points which realise it unique?

   (ii) Show that two non-parallel lines have a unique common perpendicular. [*Hint: consider separately the cases in which the minimum distance between the two lines is zero or is strictly positive.*]

7. (i) Show that every orientation-preserving isometry of hyperbolic 3-space is the product $R_1 \circ R_2$ of two *Möbius maps* of order 2.

   (ii) In terms of the geometry of the R_i, how is the division into types (elliptic / parabolic / hyperbolic / loxodromic) realised?

8. An *invariant disc* for a Kleinian group $G \leq$ Möb is a disc in $\mathbb{C} \cup \{\infty\}$ mapped to itself by every element of G.

   (i) Show that if G contains a loxodromic element it has no invariant disc.

   (ii) Give an example of a subgroup G of the Möbius group generated by two elements which contains no loxodromic element and which has no invariant disc.

   (iii) Show the limit set of G is contained in the boundary of any invariant disc.

9. Let G be a Kleinian group which contains a hyperbolic or loxodromic element. Show that the largest open set $\Omega(G)$ in $\mathbb{C} \cup \{\infty\}$ on which G acts properly discontinuously is the complement of the limit set, i.e. $\Omega(G) = (\mathbb{C} \cup \{\infty\}) \backslash \Lambda(G)$. Deduce that $\Omega(G)$ is empty if and only if all the orbits of G on the sphere are dense.

   Give an example of a Kleinian group for which the limit set is empty.

10. (i) Show that any Kleinian group G is countable.

    (ii) Show that a non-empty closed subset of a complete metric space such that every point is an accumulation point is necessarily uncountable.

    (iii) Let $G = \langle A, B \rangle$ be a free Fuchsian group associated to a triple of pairwise disjoint geodesics $\gamma_i \subset \mathbb{D} \cong \mathbb{H}^2$. Show that one can (uniquely) label the points of the limit set by (infinite length) words in $\{A, B, A^{-1}, B^{-1}\}$ but that one cannot label the points by elements of G.

    (iv) Identify the quotient space $\mathbb{H}^2/\langle A, B \rangle$. [If you know some algebraic topology, prove directly that the quotient space has fundamental group a free group of rank 2.] If G' is the larger group generated by the inversions in the γ_i, what is $\mathbb{H}^2/G'$?

11. (i) Suppose four circles lie in a tangent chain (i.e. C_i is tangent to C_{i+1} and no others for $i = 0, 1, 2, 3$ with indices mod 4). Show the four tangency points lie on a circle.

    (ii) Show two triples of pairwise tangent circles are equivalent under the action of the Möbius group. Deduce that the Appollonian gasket is conformally unique.

12. Prove the "trace identity" $tr(AB) + tr(AB^{-1}) = tr(A)tr(B)$ for matrices A and B in $SL_2(\mathbb{C})$. Deduce that traces of all words in A and B and their inverses (i.e. of all elements of the group $\langle A, B \rangle$ generated by A and B) are determined by the three numbers $\{tr(A), tr(B), tr(AB)\}$.

    (ii)* Suppose A and B are both loxodromic. Prove that $\langle A, B \rangle$ is conjugate to a subgroup of $SL_2(\mathbb{R})$ (hence defines a Fuchsian group) if and only if $(tr(A), tr(B), tr(AB)) \in \mathbb{R}^3$.

Ivan Smith
is200@cam.ac.uk