Geometry & Groups, 2014 – Sheet 2

1. Let C_1 and C_2 be two circles in $\mathbb{C} \cup \{\infty\}$.

   (i) Show there is some Möbius map taking C_1 to C_2.

   (ii) If C_1 and C_2 are disjoint, show there is a Möbius map taking the C_i to two concentric circles in $\mathbb{C}$ each centred on the origin. Hence show there is a Möbius map exchanging C_1 and C_2.

   (iii) Show that the map $z \mapsto z + \frac{1}{z}$ does not preserve the set of circles and lines in $\mathbb{C} \cup \{\infty\}$.

2. For $g \in$ Möb let $S_n(g)$ be the set of n-th roots $\{h \in \text{Möb} \,|\, h^n = g\}$.

   (i) If $g \in$ Möb satisfies $g^n(z) = z$ for some $n \geq 2$ then show g is elliptic.

   (ii) Show $g = e \Rightarrow |S_n(g)| = \infty$;

   (iii) Show that if g is parabolic $\Rightarrow |S_n(g)| = 1$;

   (iv) Show that in all other cases, $|S_n(g)| = n$.

3. Let g be a Möbius transformation and suppose z is a fixed point of g, so $g(z) = z$. Describe the set $Z(g)$ of all Möbius transformations that commute with g, and hence describe the set $\{h(z) \,|\, h \in Z(g)\}$.

4. (i) For points $p, q \in \mathbb{C}$, draw a picture of all the possible circles Γ for which inversion in Γ exchanges p and q.

   (ii) Prove that every Möbius map is a composition of inversions. How many do you need?

   (iii) If g is an elliptic isometry of the hyperbolic plane which leaves a circle C invariant, show inversion in C exchanges the two fixed points of g.

   (iv) Prove that inversion in a circle $\Gamma \subset \mathbb{C} \cup \{\infty\}$ with centre c and radius r takes a point $p \in \mathbb{C}$ to the unique point p' on the line through c and p for which $|c - p|.|c - p'| = r^2$. Deduce that inversion J_Γ in Γ is given by $J_\Gamma(z) = c + \frac{r^2}{\bar{z} - \bar{c}}$.

   (v) Let J denote inversion in the unit sphere $S^2 \subset \mathbb{R}^3$. If Σ is any sphere in $\mathbb{R}^3$, show that Σ is orthogonal to S^2 if and only if the inversions in S^2 and Σ commute, i.e. $J \circ J_\Sigma = J_\Sigma \circ J$.

5. (i) Let $S^2 \subset \mathbb{R}^3$ denote the unit sphere. Find a formula for stereographic projection $S^2 \backslash \{(0,0,1)\} \to \mathbb{C}$. Hence, or otherwise, show that the antipodal map which sends a point $(x, y, z) \in S^2$ on the sphere to its opposite $(-x, -y, -z)$ corresponds under stereographic projection to the map $J : z \mapsto -1/\bar{z}$.

   (ii) Let g be a Möbius map which preserves the usual distance on the sphere S^2 (i.e. the Euclidean distance induced by considering $S^2 \subset \mathbb{R}^3$). Show that g commutes with the map J, and hence prove that g can be represented by a matrix belonging to the group

$$SU(2) = \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \;\middle|\; |a|^2 + |b|^2 = 1 \right\}$$

(iii) Given any element of $SU(2)$, show that the corresponding Möbius map defines a rotation of S^2, and conclude that the image $\mathbb{P}SU(2) = SU(2)/\pm I$ of $SU(2)$ in the Möbius group is isomorphic to the rotation group $SO(3)$.

6. (i) Show that there is an isometry of the hyperbolic plane $\mathbb{H}^2$ taking points (p, q) to points (u, v) iff $d_{hyp}(p, q) = d_{hyp}(u, v)$.

   (ii) In the upper half-plane model $\mathfrak{h}$ of $\mathbb{H}^2$, find the centre of a hyperbolic circle of radius ρ with Euclidean centre $ic \in \mathfrak{h}$.

   (iii) Are two hyperbolic triangles of the same area in $\mathbb{H}^2$ necessarily isometric (i.e. is there an isometry taking one to the other)?

   (iv) Show that for any $n \geq 5$ $\mathbb{H}^2$ contains a regular n-gon with all interior angles being right-angles. [Hint: use a "continuity" argument to get the angles right.]

   (v) Compute the area of a regular hyperbolic hexagon all of whose interior angles are right-angles.

   (vi) If $\mathbb{H}^2$ is tessellated by compact (i.e. closed and bounded) pairwise-isometric tiles, show that the number of tiles "k steps" away from a given tile grows exponentially with k. What is the corresponding Euclidean statement?

7. (i) Let $\gamma_1, \gamma_2, \gamma_3$ be pairwise disjoint geodesics in the hyperbolic disc D, whose end-points are cyclically ordered so as to bound a "triangular" region. Let J_i denote inversion in γ_i and $A = J_2 \circ J_1$, $B = J_3 \circ J_2$. Explain why the group $\langle A, B \rangle$ is a free group.

   (ii) If the geodesics γ_i instead pairwise intersect and bound a closed triangle in D, and if $\langle A, B \rangle$ is still discrete in $\mathrm{Isom}^+(\mathbb{H}^2)$, can it still be a free group? Justify your answer.

   (iii) By exhibiting a suitable tessellation, or otherwise, prove that there is a Fuchsian group, i.e. discrete subgroup of $\mathrm{Isom}^+(\mathbb{H}^2)$, which acts on $\mathbb{H}^2$ with fundamental domain an octagon. [Hint: Escher.]

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