Geometry & Groups, 2014 — Sheet 1

1. When are two rotations conjugate in the group of orientation-preserving isometries of the Euclidean plane? What about in the group of all isometries? Justify your answers.

2. Show that $\mathbb{R}$ acts on the plane $\mathbb{R}^2$ via $t \cdot (x, y) = (e^t x, e^{-t} y)$. Draw the orbits, and find the stabilisers of points.

3. (i) Use the "orbit-stabiliser theorem" to compute the symmetry group of a cube.

   (ii) By considering a suitable pair of embedded tetrahedra, or otherwise, show that this group has a natural homomorphism onto $\mathbb{Z}/2$. Describe explicitly a non-trivial element of the kernel.

4. Let s_n denote the side length of a regular polygon with n sides, inscribed in the unit circle. Show that $s_{2n} = \sqrt{2 - \sqrt{4 - s_n^2}}$ and deduce

$$s_{2^n} = \sqrt{2 - \sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots + \sqrt{2}}}}}$$

   By considering area, deduce that

$$\pi = \lim_{n\to\infty} 2^n \sqrt{2 - \sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots + \sqrt{2}}}}}$$

   (where the final expression has n nested square roots).

5. (i) Show that the golden ratio $\tau = \frac{1+\sqrt{5}}{2}$ satisfies $\tau^2 = 1 + \tau$.

   (ii) Let P be a pentagon with side length l. Show that a diagonal joining two non-adjacent vertices of P has length τl.

   (iii) Cut each of two regular pentagons of side length 2 along such diagonals. Show that the resulting four pieces can be combined to make a "tent" of height 1 with base a square of side length 2τ. By attaching such pyramids to the faces of a cube, show that there is a dodecahedron with vertices

$$(0, \pm 1, \pm\tau^2),\ (\pm 1, \pm\tau^2, 0),\ (\pm\tau^2, 0, \pm 1),\ (\pm\tau, \pm\tau, \pm\tau).$$

   (iv) Show that the dodecahedron contains 5 such cubes, and hence prove that its full symmetry group is a subgroup of $O(3)$ isomorphic to the group $A_5 \times \mathbb{Z}_2$. Is this group isomorphic to the symmetric group S_5?

6. Draw pictures representing two different non-abelian two-dimensional Euclidean crystallographic groups ("wallpaper groups"), listing all the symmetries of the pictures.

7. Let $\Lambda \subset \mathbb{R}^2$ be a rank two lattice. A *basis* for Λ is a pair of vectors w_1, w_2 for which $\Lambda = \mathbb{Z}w_1 \oplus \mathbb{Z}w_2$. If $\{w_1, w_2\}$ and $\{w_1', w_2'\}$ are two bases for Λ, show that one can write
$$w_1' = aw_1 + bw_2 \quad \text{and} \quad w_2' = cw_1 + dw_2$$
for a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{Z})$ (note any such matrix has determinant ± 1).

8. Consider the two isometries of the Euclidean plane
$$(x, y) \mapsto (x, y + 1); \qquad (x, y) \mapsto (x + 1, -y)$$
Show (i) these generate a non-abelian group; (ii) this group acts properly discontinuously on the plane, meaning around any point (x, y) there is an open set $U_{(x,y)}$ whose images by elements $g \neq e$ are all disjoint from $U_{(x,y)}$. Find a fundamental domain for the action, and identify the quotient.

9. (i) Show that every element of $O(3)$ is a product of reflections. How many do you need? Explain why "most" elements of determinant -1 are not reflections.

   (ii) Show that every isometry of Euclidean space $\mathbb{R}^3$ with no fixed point is either a translation, a glide reflection (i.e. reflection followed by translation in a vector parallel to the plane of reflection), or a screw rotation (i.e. rotation followed by a translation parallel to the axis of rotation).

10. (i) Show that every group is a subgroup of a permutation group.

    (ii) Show that every finite group G is a subgroup of the orthogonal group $O(|G|)$.

    [Hint: *define a vector space* $\mathbb{R}^{|G|}$ *of real-valued functions on* G*. Now look at a natural action of* G *on this in an obvious basis.*]

11. Show that the space of all unoriented lines in the Euclidean plane is naturally parametrised by a Möbius band (without its boundary).

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