Geometry & Groups, Part II: 2008-9: Sheet 3

1. Let J denote inversion in the unit sphere $S^2 \subset \mathbb{R}^3$. If Σ is any sphere in $\mathbb{R}^3$, show that Σ is orthogonal to S^2 if and only if the inversions in S^2 and Σ commute, i.e. $J \circ \iota_\Sigma = \iota_\Sigma \circ J$.

2. Suppose $G \leq \text{Möb}(\mathbb{D})$ is discrete and acts properly discontinuously in $\mathbb{D}$. How are the fundamental domains for G acting on $\mathbb{D}$ and for G acting on $\mathbb{H}^3$ related?

3. Compute the area of a regular hyperbolic hexagon all of whose interior angles are right-angles.

4. The *translation length* $\eta(g)$ of an orientation-preserving isometry $g \in \text{Isom}^+(\mathbb{H}^3)$ is $\inf_{p \in \mathbb{H}^n} d_{hyp}(p, g(p))$.

   (i) Show that this is achieved along the axis of the isometry, if there is an axis. What happens if there is no axis?

   (ii) Show the translation length of $m_k : z \mapsto kz$ is $log|k|$. Find the translation length of $z \mapsto \frac{2z+1}{5z+3}$.

5. An invariant disc for a Kleinian group $G \leq$ Möb is a disc in $\mathbb{C} \cup \{\infty\}$ mapped to itself by every element of G.

   (i) Show that if G contains a loxodromic element it has no invariant disc.

   (ii) Give an example of a 2-generator subgroup G of the Möbius group which contains no loxodromic element and which has no invariant disc.

   (iii) Show the limit set of G is contained in the boundary of any invariant disc.

6. (i) Show that every orientation-preserving isometry of hyperbolic 3-space is the product $R_1 \circ R_2$ of two *Möbius maps* of order 2 (i.e. of two elliptic involutions).

   (ii) In terms of the geometry of the R_i, how is the elliptic / parabolic / loxodromic trichotomy realised?

7. Let G be a Kleinian group which contains a hyperbolic or loxodromic element. Show that the largest open set $\Omega(G)$ in $\mathbb{C} \cup \{\infty\}$ on which G acts properly discontinuously is the complement of the limit set, i.e. $\Omega(G) = (\mathbb{C} \cup \{\infty\}) \backslash \Lambda(G)$. Deduce that $\Omega(G)$ is empty if and only if all the orbits of G on the sphere are dense.

   Give an example of a Kleinian group for which the limit set is empty.

8. (i) Show that any Kleinian group is countable.

   (ii) Show that a non-empty closed subset of a complete metric space such that every point is an accumulation point is necessarily uncountable.

   (iii) What does this say about labelling points of a limit set by words in some fixed set of generators of a Kleinian group?

9. (i) Suppose four circles lie in a tangent chain (i.e. C_i is tangent to C_{i+1} and no others for $i = 0, 1, 2, 3$ with indices mod 4). Show the four tangency points lie on a circle.

   (ii) Show two triples of pairwise tangent circles are equivalent under the action of the Möbius group. Deduce that the Appollonian gasket is conformally unique.

10. The *modular group* $\mathbb{P}SL_2(\mathbb{Z})$ is a famous discrete subgroup of the Möbius group. By considering the actions of the elements $z \mapsto z+1$ and $z \mapsto -1/z$, or otherwise, show that a fundamental domain for its action on the upper half-plane $\mathfrak{h}$ is given by $\{z \in \mathfrak{h} \mid |z| > 1, Re(z) \in [-1/2, 1/2]\}$. What does the quotient $\mathfrak{h}/\mathbb{P}SL_2(\mathbb{Z})$ look like ?

11.* (i) Prove the "trace identity" $tr(AB) + tr(AB^{-1}) = tr(A)tr(B)$ for matrices A and B in $SL_2(\mathbb{C})$. Deduce that traces of all words in A and B and their inverses (i.e. of all elements of the group $\langle A, B\rangle$ generated by A and B) are determined by the three numbers $\{tr(A), tr(B), tr(AB)\}$.

   (ii) Suppose A and B are both loxodromic. Prove that $\langle A, B\rangle$ is conjugate to a subgroup of $SL_2(\mathbb{R})$ if and only if $(tr(A), tr(B), tr(AB)) \in \mathbb{R}^3$.

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