Geometry & Groups, Part II (2008-9): Sheet 2

1. Let C_1 and C_2 be two circles in $\mathbb{C} \cup \{\infty\}$.

   (i) Show there is some Möbius map taking C_1 to C_2.

   (ii) If C_1 and C_2 are disjoint, show there is a Möbius map taking the C_i to two concentric circles in $\mathbb{C}$ each centred on the origin.

   (iii) Hence, or otherwise, show there is always a Möbius map exchanging C_1 and C_2.

   (iv) Show that the map $z \mapsto z + \frac{1}{z}$ does not preserve the set of circles and lines in $\mathbb{C} \cup \{\infty\}$.

2. For $g \in$ Möb let $S_n(g)$ be the set of n-th roots $\{h \in \text{Möb} \,|\, h^n = g\}$.

   (i) If $g \in$ Möb satisfies $g^n(z) = z$ for some $n \geq 2$ then show g is elliptic.

   (ii) Show $g = e \Rightarrow |S_n(g)| = \infty$;

   (iii) Show that if g is parabolic $\Rightarrow |S_n(g)| = 1$;

   (iv) Show that in all other cases, $|S_n(g)| = n$.

3. Let A be a Möbius transformation and suppose z is a fixed point of A, so $A(z) = z$. Describe the set $Z(A)$ of all Möbius transformations that commute with A, and hence describe the set $\{B(z) \,|\, B \in Z(A)\}$.

4. Every Möbius map is a composition of inversions. How many do you need?

5. Show that the Möbius maps preserving the unit disc form the group

$$SU_{1,1} = \left\{ \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \,|\, |a|^2 - |b|^2 = 1 \right\}$$

   of 2×2 complex matrices which preserve the indefinite form $(z, w) \mapsto |z|^2 - |w|^2$. By considering $(a/|a|, b/a)$, or otherwise, show that this space of matrices is homeomorphic to an open solid torus (donut minus icing, or bagel minus sesame seeds) $\mathbb{S}^1 \times D^2$. What can you say about the topology of $SL_2(\mathbb{R})$?

6. (i) Show that there is an isometry of the hyperbolic plane taking points (p, q) to points (u, v) iff $d_{hyp}(p, q) = d_{hyp}(u, v)$.

   (ii) Show that the hyperbolic plane contains a regular pentagon with all interior angles being right-angles. [Hint: use a "continuity" argument to get the angles right.]

   (iii) If the hyperbolic plane is tessellated by compact tiles, show that the number of tiles "k steps" away from a given tile grows exponentially with k. What is the corresponding Euclidean statement?

7. (i) In the upper half-plane model, show that the distance from ip to iq, with $p < q$, is $log(q/p)$.

   (ii) Show that a hyperbolic circle is a Euclidean circle. (Does it matter whether we work in the upper half-plane or the disk to answer this?) Find the area of a hyperbolic circle with hyperbolic radius ρ. Do the Euclidean centre and the hyperbolic centre of a circle in hyperbolic space always co-incide?

8. (i) Two (distinct) hyperbolic geodesics are *parallel* if they meet at infinity. Show that two hyperbolic geodesics in hyperbolic 3-space have a unique common perpendicular if and only if they are not parallel.

   (ii) Are two hyperbolic triangles of the same area in the hyperbolic plane necessarily isometric (i.e. is there an isometry taking one to the other)?

   (iii) By working in the disk model, show that the space of oriented geodesics in the hyperbolic plane has a natural flat Euclidean structure. What about the space of oriented geodesics in hyperbolic 3-space?

9. (i) Let $\gamma \subset \mathbb{H}^3$ be a hyperbolic geodesic. Draw a rough picture of the region $\{x \in \mathbb{H}^3 \mid d(x,\gamma) < 1\}$ and observe that (in the 3-ball model, if γ does not pass through the origin) it resembles a banana.

   (ii) Find an orientation-preserving isometry of $\mathbb{H}^3$ which leaves more than one line invariant (and is not the identity!).

   (iii) Show that an orientation-preserving isometry of hyperbolic 3-space has at most one axis, *without* using the theorem that $\mathrm{Isom}^+(\mathbb{H}^3) = \text{Möb}$.

10.* This uses complex analysis. Show that the Möbius group is the group of all holomorphic automorphisms of the Riemann sphere $\mathbb{C} \cup \{\infty\}$. [Hint: if g is a holomorphic automorphism fixing 0 and ∞, consider $g(z)/z$.]

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