Geometry & Groups, Part II: 2008-9: Sheet 1

1. When are two rotations conjugate in the group of orientation-preserving isometries of the Euclidean plane? What about in the group of all isometries? Justify your answers.

2. Show that $\mathbb{R}$ acts on the plane $\mathbb{R}^2$ via $t \cdot (x, y) = (e^t x, e^{-t} y)$. Draw the orbits, and find the stabilisers of points.

3. (i) State and prove the "orbit-stabiliser theorem".

   (ii) Use it to compute the order of the full symmetry group of a cube.

   (iii) By considering a suitable pair of embedded tetrahedra, or otherwise, show that the group of rotational symmetries of a cube has a natural homomorphism onto $\mathbb{Z}/2$. Describe explicitly a non-trivial element of the kernel.

4. Consider the two isometries of the Euclidean plane

$$(x, y) \mapsto (x, y + 1); \qquad (x, y) \mapsto (x + 1, -y)$$

   Show (i) these generate a non-abelian group; (ii) this group acts properly discontinuously on the plane, meaning around any point (x, y) there is an open set $U_{(x,y)}$ whose images by elements $g \neq e$ are all disjoint from $U_{(x,y)}$. Find a fundamental domain for the action, and identify the quotient.

5. (i) Draw pictures representing five different Euclidean crystallographic groups, listing all the symmetries of the pictures.

   (ii) Show that every element of $O(3)$ is a product of reflections. How many do you need? Explain why "most" elements of determinant -1 are not reflections.

6. Let s_n denote the side length of a regular polygon with n sides, inscribed in the unit circle. Show that $s_{2n} = \sqrt{2 - \sqrt{4 - s_n^2}}$ and deduce

$$s_{2^n} = \sqrt{2 - \sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots + \sqrt{2}}}}}$$

   Deduce that

$$\pi = \lim_{n \to \infty} 2^n \sqrt{2 - \sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots + \sqrt{2}}}}}$$

   (where the final expression has n nested square roots).

7. Show that the space of all (unoriented) lines in the Euclidean plane is naturally parametrised by a Möbius band.

8.* (i) Show that every group is a subgroup of a permutation group.

(ii) Show that every finite group G is a subgroup of the orthogonal group $O(|G|)$.

[Hint: *define a vector space* $\mathbb{R}^{|G|}$ *of real-valued functions on* G*. Now look at a natural action of* G *on this in an obvious basis.*]

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