**Geometry & Groups, Part II (2007-8): Sheet 4**

1. Let $G$ be a Kleinian group which contains a hyperbolic or loxodromic element. Show that the largest open set in $\mathbb{C} \cup \{\infty\}$ on which $G$ acts properly discontinuously is the complement of the limit set $\Omega(G) = (\mathbb{C} \cup \{\infty\}) \backslash \Lambda(G)$. Deduce that $\Omega(G)$ is empty if and only if all the orbits of $G$ on the sphere are dense.

2. Let $(X, d)$ be a metric space and $\cdots \supset X_n \supset X_{n+1} \supset \cdots$ be a sequence of decreasing non-empty compact subsets of $X$. Prove that the intersection $\cap_k A_k$ is non-empty and compact. If the $A_k$ were non-empty and open would the intersection necessarily be (i) non-empty (ii) open ?

3. Show that Hausdorff distance $d_{Haus}(A, B)$ defines a metric space structure on the set of compact subsets of a given metric space.

4. (i) Find two similarities $S_1$, $S_2$ of $\mathbb{R}$ such that the unit interval $[0, 1]$ is the unique non-empty compact invariant set for the $S_i$.
   (ii) Write the Cantor set $C$ as the invariant set of a collection of *three* similarities of $\mathbb{R}$, and hence (re-)compute its Hausdorff dimension.

5. Let $F$ be a finite subset of $\mathbb{R}^n$. Show that the zero-dimensional Hausdorff measure $\mathcal{H}^0(F)$ is the cardinality of $F$.

6. Let $f : \mathbb{R} \to \mathbb{R}$ be the map $x \mapsto x^2$. Show that $\dim_H(f(F)) = \dim_H(F)$ for every subset $F \subset \mathbb{R}$.

7. (i) Compute the Hausdorff dimension of the Sierpinski carpet, given by cutting a square into nine equal pieces, and removing the central one.

   (ii) Let $F = \{x \in \mathbb{R} \,|\, x = b_m b_{m-1} \ldots b_1 . a_1 a_2 \ldots \text{ with } b_i, a_j \neq 5\}$ be those points on the line which admit decimal expansions omitting the number 5. What is $\dim_H(F)$?

   (iii) Construct a fractal in the plane whose Hausdorff dimension is given by the positive real solution $s$ to the equation $4(\frac{1}{4})^s + (\frac{1}{2})^s = 1$.

8. Show that for every $s \in [0, 2]$ there is a totally disconnected subset $F \subset \mathbb{R}^2$ for which $\dim_H(F) = s$.

9. Show that two triples of mutually tangent circles in $\mathbb{C} \cup \{\infty\}$ are conjugate under the action of the Möbius group. Deduce that the degenerate Schottky group called the Appollonian gasket is unique up to conjugacy. (Its Hausdorff dimension is therefore some universal constant, emerging from the geometry of hyperbolic 3-space.)

10. Give explicit examples of Kleinian groups realising 3 different values of Hausdorff dimension for their limit sets. Justify your answer!

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