Geometry & Groups, Part II (2007-8): Sheet 3

1. Let J denote inversion in the unit sphere $S^2 \subset \mathbb{R}^3$. If Σ is any sphere in $\mathbb{R}^3$, show that Σ is orthogonal to S^2 if and only if the inversions in S^2 and Σ commute, i.e. $J \circ \iota_\Sigma = \iota_\Sigma \circ J$.
2. Show that if a smooth homeomorphism of hyperbolic space takes geodesics to geodesics then it is an isometry. Is the same true in Euclidean space?
3. Prove that an orientation-preserving isometry of hyperbolic 3-space has at most one axis *without* using the characterisation of isometries as Möbius transformations.
4. An invariant disc for a Kleinian group $G \leq$ Möb is a disc in $\mathbb{C} \cup \{\infty\}$ mapped to itself by every element of G. (i) Show that if G contains a loxodromic element it has no invariant disc. (ii) Give an example of a 2-generator subgroup G of the Möbius group which contains no loxodromic element and which has no invariant disc. (iii) Show the limit set of G is contained in the boundary of any invariant disc.
5. Suppose $G \leq \text{Möb}(\mathbb{D})$ is discrete and acts properly discontinuously in $\mathbb{D}$. How are the fundamental domains for G acting on $\mathbb{D}$ and for G acting on $\mathbb{H}^3$ related?
6. Give an example of a Kleinian group for which the limit set is empty.
7. Prove the "trace identity" $tr(AB) + tr(AB^{-1}) = tr(A)tr(B)$ for matrices A and B in $SL_2(\mathbb{C})$. Deduce that traces of all words in A and B and their inverses (i.e. of all elements of the group generated by A and B) are determined by the three numbers $\{tr(A), tr(B), tr(AB)\}$.
8. The *modular group* $\mathbb{P}SL_2(\mathbb{Z})$ is a famous discrete subgroup of the Möbius group. By considering the actions of the elements $z \mapsto z+1$ and $z \mapsto -1/z$, or otherwise, find a fundamental domain for its action on the upper half-plane $\mathfrak{h}$. What does the quotient $\mathfrak{h}/\mathbb{P}SL_2(\mathbb{Z})$ look like ?
9. Show that any Kleinian group is countable.
10. Show the translation length of $m_k : z \mapsto kz$ is $log|k|$. Find the translation length of $z \mapsto \frac{2z+1}{5z+3}$.
11. Show that a non-empty closed subset of a complete metric space such that every point is an accumulation point is necessarily uncountable.
12. Suppose four circles lie in a tangent chain (i.e. C_i is tangent to C_{i+1} and no others for $i = 0, 1, 2, 3$ with indices mod 4). Show the four tangency points lie on a circle.

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