Geometry & Groups, Part II (2007-8): Sheet 2

1. Prove that inversion in a circle $C \subset \mathbb{C} \cup \{\infty\}$ with centre c and radius r takes a point $p \in \mathbb{C}$ to the unique point p' on the line through c and p for which $|c-p|.|c-p'| = r^2$.
2. Using the previous question, or otherwise, show that inversion in a circle in $\mathbb{C} \cup \{\infty\}$ takes circles to circles.
3. If $g \in$ Möb satisfies $g^n(z) = z$ for some $n \geq 2$ then show g is elliptic.
4. Let C_1 and C_2 be disjoint circles in $\mathbb{C} \cup \{\infty\}$. Show there is a Möbius map taking the C_i to two concentric circles in $\mathbb{C}$ each centred on the origin.
5. For $g \in$ Möb let $S_n(g)$ be the set of n-th roots $\{h \in \text{Möb} \,|\, h^n = g\}$. Show (i) $g = e \Rightarrow |S_n(g)| = \infty$; (ii) g parabolic $\Rightarrow |S_n(g)| = 1$; (iii) in all other cases, $|S_n(g)| = n$.
6. Show that the hyperbolic plane contains a regular pentagon with all interior angles being right-angles.
7. Show that there is an isometry of the hyperbolic plane taking points (p, q) to points (u, v) iff $d_{hyp}(p, q) = d_{hyp}(u, v)$.
8. If g is an elliptic isometry of the hyperbolic plane which leaves a circle C invariant, show inversion in C exchanges the two fixed points of g.
9. Let T denote a triangle in the hyperbolic plane (so the sides of T are segments of geodesics). Show the 3 angle bisectors of T meet at a point, and deduce that there is an "inscribed circle" for T. Do the 3 vertices of T necessarily lie on a circle?
10. If the hyperbolic plane is tessellated by compact tiles, show that the number of tiles "k steps" away from a given tile grows exponentially with k. What is the corresponding Euclidean statement?
11. Show there is a surjective homomorphism $SU(2) \to SO(3)$ with kernel $\{\pm I\}$.
12. Show that the Möbius maps preserving the unit disc form the group

$$SU_{1,1} = \left\{ \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \mid |a|^2 - |b|^2 = 1 \right\}$$

of 2×2 complex matrices which preserve the indefinite form $(z, w) \mapsto |z|^2 - |w|^2$. Deduce that $SL_2(\mathbb{R})$ is homeomorphic to an open solid torus (donut minus icing, or bagel minus sesame seeds) $\mathbb{S}^1 \times D^2$.

13. Find an orientation-preserving isometry of $\mathbb{H}^3$ which leaves more than one line invariant (and is not the identity!).

14. Let $\gamma \subset \mathbb{H}^3$ be a hyperbolic geodesic. Draw the region $\{x \in \mathbb{H}^3 \mid d(x, \gamma) < 1\}$ and observe that (in the 3-ball model, if γ does not pass through the origin) it resembles a banana.

15. NB: This is an optional extra, way off syllabus. Show the space of rank 2 lattices in $\mathbb{R}^2$ is $GL_2(\mathbb{R})/GL_2(\mathbb{Z})$ and that the space of rank 2 area 1 lattices is $SL_2(\mathbb{R})/SL_2(\mathbb{Z})$.

    Hard fact: A lattice determines a quotient space $\mathbb{C}/\Lambda$, which is the same holomorphically as $\{(x : y : z) \in \mathbb{CP}^2 \mid y^2 z = 4x^3 - axz^2 - bz^3\}$ for some uniquely determined $(a, b) \in \mathbb{C}^2$ for which $a^3 - b^2 \neq 0$.

    Deduce the quotient space $SL_2(\mathbb{R})/SL_2(\mathbb{Z})$ can be identified with $\mathbb{S}^3 \backslash \kappa$, the complement of a *trefoil knot* $\kappa \subset \mathbb{S}^3$.

    Note: *This is the most common knot in garden hoses, and is the unique knot which you can draw in the plane which is non-trivial and has exactly 3 crossings. More relevant to this question, if you draw a curve on a torus $\mathbb{T}^2$ which wraps around twice in one direction and three times in another, you've drawn a trefoil in space. Now think about $\mathbb{S}^3 \subset \mathbb{C}^2$ and look at that condition on (a, b) again.*

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