Geometry & Groups, Part II: 2007-8: Sheet 1

1. When are two rotations conjugate in the group of orientation-preserving isometries of the Euclidean plane? What about in the group of all isometries? Justify your answers.

2. Show that $\mathbb{R}^* = \mathbb{R}\backslash\{0\}$ acts on the plane $\mathbb{R}^2$ via $t \cdot (x,y) = (e^t x, e^{-t} y)$. Draw the orbits, and find the stabilisers of points.

3. Use the orbit-stabiliser theorem to compute the order of the full symmetry group of a cube.

4. By considering a suitable pair of embedded tetrahedra, or otherwise, show that the group of rotational symmetries of a cube has a natural homomorphism onto $\mathbb{Z}/2$. Describe explicitly a non-trivial element of the kernel.

5. Consider the two isometries of the Euclidean plane
$$(x,y) \mapsto (x, y+1); \qquad (x,y) \mapsto (x+1, -y)$$
Show (i) these generate a non-abelian group; (ii) this group acts properly discontinuously on the plane. Find a fundamental domain for the action, and identify the quotient.

6. Let $\Lambda \subset \mathbb{R}^2$ be a lattice. Let w_1 be non-zero of minimal length in Λ and w_2 be of minimal length in $\Lambda\backslash\mathbb{Z}\langle w_1\rangle$. Show that $\Lambda = \mathbb{Z}\langle w_1, w_2\rangle$.

7. Draw pictures representing five different Euclidean crystallographic groups, explaining the symmetries of the pictures and hence describing all the elements of these groups.

8. Show that every element of $O(3)$ is a product of reflections. How many do you need? Explain why "most" elements of determinant -1 are not reflections.

9. Show that the space of all (unoriented) lines in the Euclidean plane is naturally parametrised by a Möbius band.

10. Let s_n denote the side length of a regular polygon with n sides, inscribed in the unit circle. Show that $s_{2n} = \sqrt{2 - \sqrt{4 - s_n^2}}$ and deduce
$$s_{2^n} = \sqrt{2 - \sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots + \sqrt{2}}}}}$$
By considering areas, deduce that
$$\pi = \lim_{n\to\infty} 2^n \sqrt{2 - \sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots + \sqrt{2}}}}}$$
(where the final expression has n nested square roots).

11. (i) Show that every group is a subgroup of a permutation group. (ii) Show that every finite group G is a subgroup of the orthogonal group $O(|G|)$. [Hint: *define a vector space* $\mathbb{R}^{|G|}$ *of real-valued functions on* G*. Now look at a natural action of* G *on this in an obvious basis.*]

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