GEOMETRY AND GROUPS – Example Sheet 2

TKC Michaelmas 2006

1. Two linearly independent vectors $\boldsymbol{w}_1, \boldsymbol{w}_2$ are a *basis* for a lattice Λ if $\Lambda = \mathbb{Z}\boldsymbol{w}_1 + \mathbb{Z}\boldsymbol{w}_2$. Show that the pair $\boldsymbol{w}_1', \boldsymbol{w}_2'$ are also a basis for Λ if, and only if,

$$\boldsymbol{w}_1' = a\boldsymbol{w}_1 + b\boldsymbol{w}_2$$
$$\boldsymbol{w}_2' = c\boldsymbol{w}_1 + d\boldsymbol{w}_2$$

for a matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with integer entries that has an inverse M^{-1} which also has integer entries. Prove that $ad - bc = \pm 1$.

2. Λ is a rank 2 lattice in $\mathbb{R}^2$. Choose a vector $\boldsymbol{w}_1 \in \Lambda \setminus \{\boldsymbol{0}\}$ with norm $||\boldsymbol{w}_1||$ as small as possible. Then choose $\boldsymbol{w}_2 \in \Lambda \setminus \mathbb{Z}\boldsymbol{w}_1$ with norm as small as possible. Show that $\Lambda = \mathbb{Z}\boldsymbol{w}_1 + \mathbb{Z}\boldsymbol{w}_2$.

   Let $\boldsymbol{w}_1$ be a fixed vector. Draw the region of possible values for $\boldsymbol{w}_2$. Mark on your picture the points $\boldsymbol{w}_2$ that correspond to lattices $\mathbb{Z}\boldsymbol{w}_1 + \mathbb{Z}\boldsymbol{w}_2$ that have a reflective symmetry.

3. Prove the formula for the chordal distance between two points $z_1, z_2 \in \mathbb{C} \cup \{\infty\}$ algebraically by using the formula for stereographic projection.

4. Let Γ_1, Γ_2 be two disjoint circles on the Riemann sphere. Show that there is a Möbius transformation that maps them to two circles in $\mathbb{C}$ centred on 0.

5. Find all of the Möbius transformations that commute with M_k for a fixed k. Hence describe the group

$$Z(T) = \{A \in \text{Möb} : A \circ T = T \circ A\}$$

   for an arbitrary Möbius transformation T. Describe the set $\{A(z_o) : A \in Z(T)\}$ for z_o a point in $\mathbb{P}$.

6. Suppose that the Möbius transformation T is represented by the matrix M but that $\det M \neq 1$. Show that T is parabolic if and only if $(\operatorname{tr} M)^2 = 4 \det M$. Establish similar conditions for T to be elliptic, hyperbolic or loxodromic.

7. Prove that the composition of two inversions is a Möbius transformation. Show that every Möbius transformation can be written as the composition of inversions. How many inversions do we need?

8. Show that inversion in any circle is given by a map

$$J : z \mapsto \frac{a\overline{z} + b}{c\overline{z} + d}$$

   for some complex numbers a, b, c, d with $ad - bc = 1$. For which choices of a, b, c, d is this map J an involution, that is $J^2 = I$? Are these all inversions?

9. How many square roots of a Möbius transformation are there? This means, for each Möbius transformation T, how many Möbius transformations S are there with $S^2 = T$?

10. Show that a Möbius transformation T represented by a matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is an isometry of the Riemann sphere for the chordal metric if, and only if, $M \in \mathrm{SU}(2)$. Deduce that there is a group homomorphism $\phi : \mathrm{SU}(2) \to \mathrm{SO}(3)$ with kernel $\{I, -I\}$. For each point $z_o \in \mathbb{P}$, show that there is a matrix $M \in \mathrm{SU}(2)$ with $T(0) = z_o$. Hence show that ϕ is surjective and so $\mathrm{SU}(2)/\{I, -I\} \cong \mathrm{SO}(3)$.

11. Let $\boldsymbol{p}, \boldsymbol{q}$ be two distinct points in $\mathbb{P}$. Show that there are infinitely many inversions that interchange $\boldsymbol{p}$ and $\boldsymbol{q}$. Draw a picture illustrating the circles Γ for which inversion in Γ interchanges $\boldsymbol{p}$ and $\boldsymbol{q}$.

    Now suppose that $\boldsymbol{p}'$ is another point of the Riemann sphere distinct from $\boldsymbol{p}$ and $\boldsymbol{q}$. Mark on your picture all the possible values for $J(\boldsymbol{p}')$ for inversions J that interchange $\boldsymbol{p}$ and $\boldsymbol{q}$.

    Given 4 distinct points $\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{p}', \boldsymbol{q}'$, when can we find an inversion which interchanges both $\boldsymbol{p}$ & $\boldsymbol{q}$ and also $\boldsymbol{p}'$ & $\boldsymbol{q}'$.

12. Show that there is an isometry T of $\mathbb{D}$ with the hyperbolic metric that maps z_1 and z_2 to w_1 and w_2 respectively if, and only if, $\rho(z_1, z_2) = \rho(w_1, w_2)$.

13. Show that every straight line in the Euclidean plane can be written as

$$\ell = \{t\boldsymbol{u} + \boldsymbol{v} : t \in \mathbb{R}\}$$

for $\boldsymbol{u}$ a unit vector in $\mathbb{R}^2$ and $\boldsymbol{v}$ orthogonal to $\boldsymbol{u}$. Are $\boldsymbol{u}$ and $\boldsymbol{v}$ uniquely determined by the line ℓ? Deduce that the set of lines in the Euclidean plane corresponds to the points of a Möbius band.

Is the same true for geodesics in the hyperbolic plane? (Hint: Consider the endpoints of the geodesic.) Is the same true for great circles in Riemann sphere?

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