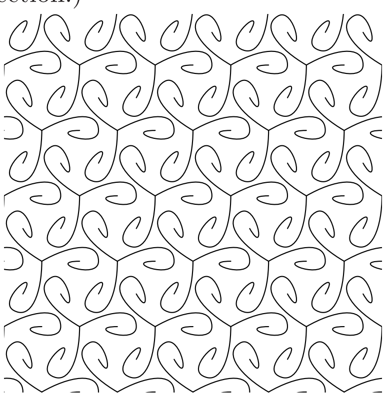
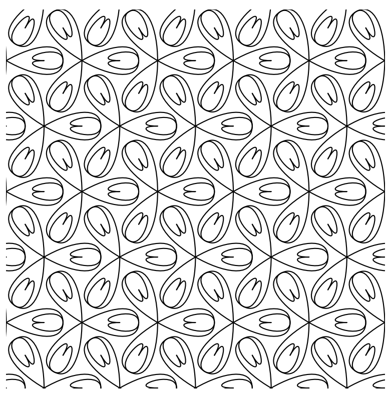
GEOMETRY AND GROUPS — Example Sheet 1

TKC Michaelmas 2006

1. Use the orbit – stabilzer theorem to compute the size of the symmetry group of a cube. Describe each of the symmetries in this group. Show that the orbit $\mathrm{Orb}(x)$ usually contains as many points as the symmetry group. Find all of the points for which this is untrue.
2. Show that additive the group $\mathbb{Z} \times \mathbb{Z}$ acts on the plane $\mathbb{R}^2$ by
$$\begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + n_1 \\ x_2 + n_2 \end{pmatrix}$$
and that the unit square $S = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : 0 \leqslant x_1 < 1 \text{ and } 0 \leqslant x_2 < 1 \right\}$ is a fundamental set. Hence show that we can identify the quotient $\mathbb{R}^2/\mathbb{Z} \times \mathbb{Z}$ with a torus.
Let $\boldsymbol{u} = \begin{pmatrix} a \\ c \end{pmatrix}$, $\boldsymbol{v} = \begin{pmatrix} b \\ d \end{pmatrix}$ for some **integers** a, b, c, d with $ad - bc = \pm 1$. Show that every vector $\boldsymbol{v} \in \mathbb{Z} \times \mathbb{Z}$ can be written as $m\boldsymbol{u} + n\boldsymbol{v}$ for some integers m and n. Deduce that the parallelogram
$$\{\lambda \boldsymbol{u} + \mu \boldsymbol{v} : 0 \leqslant \lambda < 1 \text{ and } 0 \leqslant \mu < 1\} \ .$$
is also a fundamental set for the group action.
3. Consider the two maps:
$$A : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} x_1 \\ x_2 + 1 \end{pmatrix} \ ; \quad B : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} x_1 + 1 \\ -x_2 \end{pmatrix}$$
acting on the plane $\mathbb{R}^2$. Let G be the group they generate. Is G Abelian? Find the orbit of a point $\boldsymbol{x}$ under this group. Find a fundamental set and hence describe the quotient $\mathbb{R}^2/G$.
4. Show that there are two ways to embed a regular tetrahedron in cube C so that the vertices of the tetrahedron are also vertices of C. Show that the symmetry group of C permutes these tetrahedra and deduce that the symmetry group of C is isomorphic to the Cartesian product $S_4 \times C_2$ of the symmetric group S_4 and the cyclic group C_2.
5. Show that two rotations are conjugate in $\mathrm{Isom}^+(\mathbb{E}^2)$ if and only if they are both rotations through the same angle. When are they conjugate in $\mathrm{Isom}(\mathbb{E}^2)$?
Describe all of the conjugacy classes of $\mathrm{Isom}^+(\mathbb{E}^2)$ and of $\mathrm{Isom}(\mathbb{E}^2)$.
Let $\mathcal{C}$ be the conjugacy class in $\mathrm{Isom}(\mathbb{E}^2)$ of the reflection M in a line ℓ. Show that $\mathrm{Isom}(\mathbb{E}^2)$ acts on $\mathcal{C}$ by
$$(A, R) \mapsto A \circ R \circ A^{-1} \ .$$
Identify the stablizer of M. How is this related to the stabilizer of another element $A \circ M \circ A^{-1}$ of $\mathcal{C}$?
6. Describe all of the symmetries of the two patterns below. (Both patterns are continued indefinitely in each direction.)

7. Prove Proposition 2.4 classifying the isometries of Euclidean space $\mathbb{E}^3$.

8. (Every finite group is a symmetry group.)
Let G be any finite group and let R be the set of all functions $\phi : G \to \mathbb{R}$. Show that R is a finite dimensional real vector space. Show that the group G acts on R via

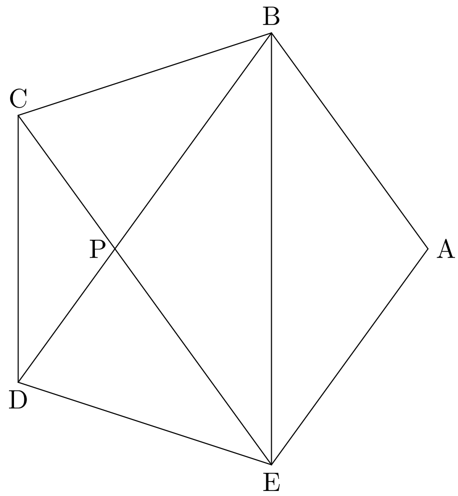
$$(g, \phi) \mapsto g \cdot \phi \qquad \text{where} \qquad g \cdot \phi : h \mapsto \phi(g^{-1}h) \ .$$

Find an inner product on R that makes the functions

$$\varepsilon_g : h \mapsto \begin{cases} 1 & \text{when } h = g; \\ 0 & \text{otherwise.} \end{cases}$$

into an orthonormal basis for R. Show that each element of G then acts as an orthogonal linear map on R.

9. The number $\tau = \frac{1}{2}(1+\sqrt{5})$ is called the *Golden ratio*. Show that it satisfies $\tau^2 = \tau + 1$.



In the diagram above, $ABCDE$ is a regular pentagon. Show that the triangles ABE, PEB and PCD are similar. Deduce that the diagonal BE has length τ times the side length for the pentagon.

10. Take two regular pentagons with sides of length 2 and cut them along a diagonal joining two non-adjacent vertices. Show that the four pieces can be fitted together to form a tent over a square with side length 2τ. Show that the height of the tent is then 1. Attach six of these tents to the faces of a cube and hence show that the twenty points

$$(0, \pm 1, \pm\tau^2), (\pm 1, \pm\tau^2, 0), (\pm\tau^2, 0, \pm 1), (\pm\tau, \pm\tau, \pm\tau)$$

are the vertices of a regular dodecahedron.
Note that the cube is inscribed inside the dodecahedron. How many such inscribed cubes are there?

11. Let s_n, $n \geqslant 3$, be the side length of a regular n-gon P_n inscribed inside the unit circle. Show that $s_{2n} = \sqrt{2 - \sqrt{4 - s_n^2}}$. Deduce that

$$s_{2^n} = \sqrt{2 - \sqrt{2 + \sqrt{2 + \ldots + \sqrt{2}}}} \ .$$

Let A_n be the area of P_n. Show that

$$A_{2^{n+1}} = 2^{n-1} s_{2^n}$$

and deduce that

$$\pi = \lim_{n\to\infty} 2^n \sqrt{2 - \sqrt{2 + \sqrt{2 + \sqrt{2 + \ldots + \sqrt{2}}}}}$$

where there are n nested square roots in the limit.

Please send any comments or corrections to me at: t.k.carne@dpmms.cam.ac.uk .