Galois Theory (Michaelmas 2005):
Transitivity of Trace and Norm

a.j.scholl@dpmms.cam.ac.uk

Theorem (14.4). *Let $M/L/K$ be finite extensions and $x \in M$. Then*

$$\mathrm{Tr}_{M/K}(x) = \mathrm{Tr}_{L/K}(\mathrm{Tr}_{M/L}(x)), \quad \mathrm{N}_{M/K}(x) = \mathrm{N}_{L/K}(\mathrm{N}_{M/L}(x))$$

In the lectures this was proved only for trace. For the general result, one way is to first prove:

Lemma. *Suppose that $M \supset L \supset K$, $[M : L] = m$ and $x \in L$. Then*

$$f_{x,M/K} = f_{x,L/K}^m, \quad \mathrm{Tr}_{M/K}(x) = m\mathrm{Tr}_{L/K}(x) \text{ and } \mathrm{N}_{M/K}(x) = \mathrm{N}_{L/K}(x)^m.$$

Proof. Choose bases $u_1, \dots, u_m$ for M/L and $v_1, \dots, v_n$ for L/K, and let A be the matrix of $T_{x,L/K}$. Then in terms of the basis $\{u_i v_j\}$ for M/K, $T_{x,M/K}$ has matrix

$$\begin{pmatrix} A & 0 & \dots & 0 \\ 0 & A & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & A \end{pmatrix}$$

so its characteristic polynomial is the m-th power of that of A. The identities for trace and norm follow at once. □

Proof of Theorem. Initially we first assume that $M = L(x)$. Let $m = [L : K]$ and let $f = X^n + a_{n-1}X^{n-1} \dots + a_0$ be the minimal polynomial of x over L. Choose a basis $e_1, \dots, e_m$ for L/K and let the matrix of T_{a_i} for this basis be A_i. Then

$$\mathrm{Tr}_{M/L}(x) = -a_{n-1}, \quad \mathrm{N}_{M/L}(x) = (-1)^n a_0$$

hence

$$\begin{aligned} \mathrm{Tr}_{L/K}(\mathrm{Tr}_{M/L}(x)) &= -\mathrm{Tr}_{L/K}(a_{n-1}) = -\mathrm{Tr}(A_{n-1}) \\ \mathrm{N}_{L/K}(\mathrm{N}_{M/L}(x)) &= (-1)^{mn}\mathrm{N}_{L/K}(a_0) = (-1)^{mn}\det(A_0) \end{aligned}$$

On the other hand, the matrix of $T_{x,M/K}$ for the basis $\{e_i x^{j-1}\}$ ($1 \le i \le m$, $1 \le j \le n$) is

$$\begin{pmatrix} 0 & 0 & \dots & 0 & -A_0 \\ I_m & 0 & \dots & 0 & -A_1 \\ 0 & I_m & \dots & 0 & -A_2 \\ \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & I_m & -A_{n-1} \end{pmatrix}$$

which has trace $-\operatorname{Tr}(A_{n-1})$. Applying a cyclic permutation of the columns to the right m times, we see that its determinant is

$$(-1)^{m(mn-1)} \begin{vmatrix} -A_0 & 0 & \dots & 0 \\ -A_1 & I_m & \dots & 0 \\ \vdots & & \ddots & \vdots \\ -A_{n-1} & 0 & \dots & I_m \end{vmatrix} = (-1)^{mn} \det(A_0)$$

Now for the general case, we consider the tower $M/L(x)/L/K$. Then

$$\begin{aligned} \operatorname{Tr}_{M/K}(x) &= [M : L(x)]\operatorname{Tr}_{L(x)/K}(x) && \text{by the Lemma} \\ &= [M : L(x)]\operatorname{Tr}_{L(x)/L}(\operatorname{Tr}_{L/K}(x)) && \text{by what we have already proved} \\ &= \operatorname{Tr}_{M/L}(\operatorname{Tr}_{L/K}(x)) && \text{by the Lemma again} \end{aligned}$$

and for norm,

$$\begin{aligned} \mathrm{N}_{M/K}(x) &= \mathrm{N}_{L(x)/K}(x)^{[M:L(x)]} \\ &= \mathrm{N}_{L(x)/L}(\mathrm{N}_{L/K}(x))^{[M:L(x)]} \\ &= \mathrm{N}_{M/L}(\mathrm{N}_{L/K}(x)). \end{aligned}$$

□