EXAMPLE SHEET 4

1. Compute the 12th cyclotomic polynomial $\Phi_{12}(t)$ over the rationals.

2. Let $L$ be the 15th cyclotomic extension of the rationals. Find all the degree two extensions of the rationals contained in $L$.

3. Let $K$ be the rationals and let $M = K(\zeta)$ be the $n$th cyclotomic field with $\zeta = e^{2\pi i/n}$. Find all the subfields of $M$ expressing them in the form $K(\alpha)$.

4. Let $\Phi_n(t)$ be the $n$th cyclotomic polynomial over the rationals. Show that
   (i) If $n$ is odd then $\Phi_{2n}(t) = \Phi_n(-t)$.
   (ii) If $p$ is a prime dividing $n$ then $\Phi_{np}(t) = \Phi_n(tp)$.
   (iii) If $p$ and $q$ are distinct primes then the coefficients of $\Phi_{pq}(t)$ are either +1, 0 or -1.
   (iv) if $n$ is not divisible by at least three distinct odd primes then the coefficients of $\Phi_n(t)$ are -1, 0 or +1.
   (v) $\Phi_{3 \times 5 \times 7}(t)$ has at least one coefficient which is not -1, 0 or +1.

5. Let $f(t)$ be an irreducible cubic polynomial over a field $K$ of characteristic $\neq 2$. Let $\Delta$ be a square root of the discriminant of $f(t)$. Show that $f(t)$ remains irreducible over $K(\Delta)$.

6. Let $f(t)$ be an irreducible separable quartic and $g(t)$ be its resolvent cubic. Show that the discriminant of $f(t)$ and $g(t)$ are the same.

7. Let $K$ be the rationals. Show that $K(\sqrt{2 + \sqrt{2 + \sqrt{2}}})$ is a Galois extension of $K$ and find its Galois group.

8. (i) Show the Galois group of $f(t) = t^5 - 4t + 2$ over the rationals $K$ is $S_5$, and determine the Galois group over $K(i)$.
   (ii) Find the Galois group of $f(t) = t^4 - 4t + 2$ over the rationals $K$ and over $K(i)$.

9. Let $G$ be the group of invertible $n \times n$ upper triangular matrices with entries in a finite field $F$. Show that $G$ is soluble.
10. Express $\sum_{i \neq j} t_i^3 t_j$ as a polynomial in the elementary symmetric polynomials.

11. Show that for any $n > 1$ the polynomial $t^n + t + 3$ is irreducible over the rationals. Determine its Galois group for $n \leq 5$.

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