II Galois Theory Michaelmas Term 2016

EXAMPLE SHEET 4

1. Compute the 12th cyclotomic polynomial $\Phi_{12}(t)$ over the rationals.

2. Let L be the 15th cyclotomic extension of the rationals. Find all the degree two extensions of the rationals contained in L.

3. Let K be the rationals and let $M = K(\zeta)$ be the nth cyclotomic field with $\zeta = e^{2\pi i/n}$. Find all the subfields of M expressing them in the form $K(\alpha)$.

4. Let $\Phi_n(t)$ be the nth cyclotomic polynomial over the rationals. Show that

(i) If n is odd then $\Phi_{2n}(t) = \Phi_n(-t)$.

(ii) If p is a prime dividing n then $\Phi_{np}(t) = \Phi_n(t^p)$.

(iii) If p and q are distinct primes then the coefficients of $\Phi_{pq}(t)$ are either +1, 0 or -1.

(iv) if n is not divisible by at least three distinct odd primes then the coefficients of $\Phi_n(t)$ are -1, 0 or +1.

(v) $\Phi_{3\times5\times7}(t)$ has at least one coefficient which is not -1, 0 or +1.

5. Let $f(t)$ be an irreducible cubic polynomial over a field K of characteristic $\neq 2$. Let Δ be a square root of the discriminant of $f(t)$. Show that $f(t)$ remains irreducible over $K(\Delta)$.

6. Let $f(t)$ be an irreducible separable quartic and $g(t)$ be its resolvent cubic. Show that the discriminant of $f(t)$ and $g(t)$ are the same.

7. Let K be the rationals. Show that $K(\sqrt{2+\sqrt{2+\sqrt{2}}})$ is a Galois extension of K and find its Galois group.

8. (i) Show the Galois group of $f(t) = t^5 - 4t + 2$ over the rationals K is S_5, and determine the Galois group over $K(i)$.

(ii) Find the Galois group of $f(t) = t^4 - 4t + 2$ over the rationals K and over $K(i)$.

9. Let G be the group of invertible $n \times n$ upper triangular matrices with entries in a finite field F. Show that G is soluble.

10. Express $\Sigma_{i \neq j} t_i^3 t_j$ as a polynomial in the elementary symmetric polynomials.

11. Show that for any $n > 1$ the polynomical $t^n + t + 3$ is irrediucible over the rationals. Determine its Galois group for $n \leq 5$.

brookes@dpmms.cam.ac.uk