(1) Let \( K \) be a finite field. By considering the multiplicative group \( K^\times \), or otherwise, write down a non-constant polynomial over \( K \) which does not have a root in \( K \). Deduce that \( K \) cannot be algebraically closed.

(2) Let \( K \) be a field and \( \overline{K} \) its algebraic closure. Assume \( K \subseteq L \) is a finite field extension. Show that \( L \) is \( K \)-isomorphic to some subfield of \( \overline{K} \).

(3) Let \( K_1 \) and \( K_2 \) be algebraically closed fields of the same characteristic. Show that either \( K_1 \) is isomorphic to a subfield of \( K_2 \) or \( K_2 \) is isomorphic to a subfield of \( K_1 \).

(4) Find an example of a field extension \( K \subseteq L \) which is normal but not separable.

(5) Let \( K \subseteq L \) be a field extension with \( [L : K] = 2 \). Show that the extension is normal.

(6) Find finite field extensions \( K \subseteq F \subseteq L \) such that \( K \subseteq F \) and \( F \subseteq L \) are normal but \( K \subseteq L \) is not normal.

(7) Let \( L \) be the splitting field of \( t^3 - 2 \) over \( \mathbb{Q} \). Find a subgroup of \( \text{Gal}(L/\mathbb{Q}) \) which is not a normal subgroup.

(8) Let \( K \subseteq L \) be a finite Galois extension, and \( F, M \) intermediate fields. What is the subgroup of \( \text{Gal}(L/K) \) corresponding to the subfield \( F \cap M \)? Show that if there is a \( K \)-isomorphism \( F \to M \), then the subgroups \( \text{Gal}(L/F) \) and \( \text{Gal}(L/M) \) are conjugate in \( \text{Gal}(L/K) \).

(9) Show that \( \mathbb{Q} \subseteq L = \mathbb{Q}(\sqrt{2}, \sqrt{-1}) \) is a Galois extension and determine its Galois group. Write down all the subgroups of \( \text{Gal}(L/\mathbb{Q}) \) and the corresponding subfields of \( L \).

(10) Show that for any natural number \( n \) there exists a Galois extension \( K \subseteq L \) with \( \text{Gal}(L/K) \) isomorphic to \( S_n \), the symmetric group of degree \( n \). Show that for any finite group \( G \) there exists a Galois extension whose Galois group is isomorphic to \( G \). (Hint: to prove the first claim, consider the field \( L = \mathbb{Q}(t_1, \ldots, t_n) \) of rational functions in \( t_1, \ldots, t_n \), then consider an action of \( S_n \) on \( L \), etc.)

(11) Let \( L \) be the splitting field of \( t^5 - 4t + 2 \) over \( \mathbb{Q} \). Show that \( \mathbb{Q} \subseteq L \) is a Galois extension with Galois group isomorphic to \( S_5 \).

(12) Let \( L \) be the splitting field of \( t^4 + t^3 + 1 \) over a field \( K \). Compute the Galois group \( \text{Gal}(L/K) \) for each of the following cases: \( K = \mathbb{F}_2 \), \( K = \mathbb{F}_3 \), and \( K = \mathbb{F}_4 \).
(13) Let \( p \) be a prime number and \( L = \mathbb{F}_p(t) \) be the field of rational functions in \( t \). Let \( a \in \mathbb{F}_p \) be a non-zero element, and let \( \varphi \in \text{Aut}_{\mathbb{F}_p}(L) \) be the automorphism determined by \( \varphi(t) = at \). Determine the subgroup \( G \leq \text{Aut}_{\mathbb{F}_p}(L) \) generated by \( \varphi \), and its fixed field \( L^G \).

(14) Show that there is at least one irreducible polynomial \( f \in \mathbb{F}_5[t] \) with \( \deg f = 17 \).

(15) Compute \( \Phi_{12} \in \mathbb{Z}[t] \), the 12-th cyclotomic polynomial.

(16) Let \( K \subseteq L \) be an extension of finite fields. Show that \( L \) is the \( n \)-th cyclotomic extension of \( K \) for some \( n \).

(17) Let \( L \) be the 7-th cyclotomic extension of \( \mathbb{Q} \). Find all the intermediate fields \( \mathbb{Q} \subseteq F \subseteq L \) and write each one as \( \mathbb{Q}(\alpha) \) for some \( \alpha \). Which one of these intermediate fields is Galois over \( \mathbb{Q} \)?

(18) Let \( \Phi_n \in \mathbb{Z}[t] \) denote the \( n \)-th cyclotomic polynomial. Show that:

(i) If \( n > 1 \) is odd, then \( \Phi_{2n}(t) = \Phi_n(-t) \).
(ii) If \( p \) is a prime dividing \( n \), then \( \Phi_{np}(t) = \Phi_n(t^p) \).
(iii) If \( p \) and \( q \) are distinct primes, then the non-zero coefficients of \( \Phi_{pq} \) are alternately +1 and −1. ([Hint: First show that if \( 1/(1 - t^p)(1 - t^q) \) is expanded as a power series in \( t \), then the coefficients of \( t^m \) with \( m < pq \) are either 0 or 1.] )
(iv) If \( n \) is not divisible by at least three distinct odd primes, then the coefficients of \( \Phi_n \) are 1, 0 or −1.
(v) \( \Phi_{105} \) has at least one coefficient which is not 1, 0 or −1.

(19) Let \( \mu = \exp(2\pi i/n) \) where \( i = \sqrt{-1} \), and let \( L = \mathbb{Q}(\mu) \) be the \( n \)-th cyclotomic extension of \( \mathbb{Q} \). Show that the isomorphism \( \text{Gal}(L/\mathbb{Q}) \to (\mathbb{Z}/(n))^\times \) sends the automorphism given by complex conjugation to the class of \( -1 \). Deduce that if \( n \geq 3 \), then \( [L : L \cap \mathbb{R}] = 2 \) and show that \( L \cap \mathbb{R} = \mathbb{Q}(\mu + \mu^{-1}) = \mathbb{Q}(\cos 2\pi/n) \).