Stochastic Financial Models – Example sheet 4
Lent 2017, SA

Problem 1. Show that the Black–Scholes price of a European call option is strictly convex in both the strike price \( K \) and the initial stock price \( S_0 \), decreasing in \( K \) and increasing in \( S_0 \). Show that the price increases with the interest rate \( r \), and with the expiry \( T \).

What are the corresponding statements for the Black–Scholes price of a European put option?

Problem 2. Consider a stock which pays a dividend at constant rate \( \delta \geq 0 \). The price of the stock is modelled by

\[
S_t = S_0 e^{\sigma W_t + (r - \delta - \sigma^2/2)t}.
\]

where \( r \geq 0 \) is the risk-free interest rate, for a Brownian motion \( W \). Show that

\[
e^{-rt}S_t + \int_0^t e^{-ru}S_u \delta du
\]
defines a martingale. Why then is \( S \) a sensible model for the stock price (at least under an equivalent measure)? Show that the time-0 value of a European put option with strike \( K \) and expiry \( T \) written on this asset is

\[
K e^{-rT} \Phi \left( \frac{\log(K/S_0) - (r - \delta + \sigma^2/2)T}{\sigma \sqrt{T}} \right) - S_0 e^{-\delta T} \Phi \left( \frac{\log(K/S_0) - (r - \delta - \sigma^2/2)T}{\sigma \sqrt{T}} \right).
\]

Can we deduce the price of the European call by put-call parity in the case when the stock pays dividends?

Problem 3. Show that the joint moment generating function of a Brownian motion and its maximum is given by

\[
E(e^{aW_t + b \max_{0 \leq s \leq t} W_s}) = \frac{2}{2a + b} \left( (a + b) \Phi(\sqrt{t}e^{(a+b)^2/2}) + a \Phi(-\sqrt{t}e^{a^2/2}) \right).
\]

Problem 4. A European lookback call option entitles the holder to buy one unit of stock at the expiry time \( T \) at the lowest price reached by the stock during the life of the option. Thus, if it is purchased at time 0, at time \( T \) it pays off the amount \( S_T - \inf_{0 \leq u \leq T} S_u \). Find the initial price of such an option in the Black-Scholes model.

Problem 5. Let \( EC(S_0, K, \sigma, r, T) \) denote the Black–Scholes price of a European call option with strike \( K \), expiry \( T \) on an asset with initial price \( S_0 \), volatility \( \sigma \), when the constant interest rate is \( r \). Show that the price of a down-and-out call with strike \( K \) and a barrier at \( B \), where \( B < \min\{S_0, K\} \), can be expressed in terms of \( EC \) as

\[
EC(S_0, K, \sigma, r, T) - (B/S_0)^{2r/\sigma^2 - 1} EC(B^2/S_0, K, \sigma, r, T).
\]
Problem 6. In the Black–Scholes model, find the time-0 prices of European contingent claims which pay at time \( T \) the amounts:

(a) \( \int_0^T S_u \, du \)

(b) \( (\log S_T)^2 \).

Problem 7. Given times \( 0 < T_0 < T_1 \), a forward start call option gives the right (but not the obligation) to buy a certain stock at time \( T_1 \) at the stock’s price at time \( T_0 \). Explain why the payout is \( (S_{T_1} - S_{T_0})^+ \) and find its initial price in the Black–Scholes model. How is this option hedged?

Problem 8. The Black–Scholes PDE is given by

\[
\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 s^2 \frac{\partial^2 V}{\partial s^2} + rs \frac{\partial V}{\partial s} - rV = 0.
\]

(i) Set \( v(\tau, x) = V(T - 2\tau/\sigma^2, e^x) \). Show that

\[
\frac{\partial v}{\partial \tau} - \frac{\partial^2 v}{\partial x^2} + (1 - k) \frac{\partial v}{\partial x} + kv = 0
\]

for constant \( k \) which you should determine.

(ii) Let \( \alpha, \beta \) be real constants and defined \( u(\tau, x) \) via

\[ u(\tau, x) = e^{\alpha x + \beta \tau} v(\tau, x). \]

Find the PDE for \( u(\tau, x) \) and choose \( \alpha, \beta \) such that \( u \) satisfies the standard heat equation,

\[ \frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}. \]

Problem 9. Consider the standard heat equation

\[ \frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}. \]

Assume \( t \in [0, T], x \in [0, L] \) and initial/boundary data

\[ u(0, x) = g(x), \quad u(t, 0) = a(t), \quad u(t, L) = b(t). \]

The grid \( \{(i \delta_t, j \delta_x) : i = 1, \ldots, N_t, j = 1, \ldots, N_x\} \) with \( \delta_x = L/N_x, \delta_t = T/N_t \) is given and we seek approximations \( U^i_j \approx u(i \delta_t, j \delta_x) \). Set

\[ U^i = \left( U^i_j \right)_{j=1}^{N_x - 1} \in \mathbb{R}^{N_x - 1}. \]

(i) Formulate the FTCS-method as linear equations

\[ U^{i+1} = FU^i + p^i \]

for some \((N_x - 1) \times (N_x - 1)\)-matrix \( F \) and \( p^i \in \mathbb{R}^{N_x - 1} \).
(ii) Similarly, formulate the BTCS-method as

$$BU^{i+1} = U^i + q^i$$

for a matrix $B$ and a vector $q^i$ to be determined.

(iii) Show that adding these two linear equations yields exactly the Crank–Nicolson method.