Problem 1. Suppose that $X_1, X_2, \ldots$ are i.i.d. real random variables with $\mathbb{E}(|X_1|) < \infty$. Let $S_0 = 0$ and $S_n = X_1 + \ldots + X_n$.

(a) When is $(S_n)_{n \geq 0}$ a martingale? Specify the filtration.

(b) Show that $\mathbb{E}[X_1 | S_n] = \frac{S_n}{n}$.

(c) Compute $\mathbb{E}[S_n | X_1]$.

(d) Find an example of a process $(Z_n)_{n \geq 0}$ adapted to some filtration which has the property $\mathbb{E}[Z_{n+1} | Z_n] = Z_n$ for all $n \geq 0$, but $\mathbb{E}[Z_{N+1} | \mathcal{F}_N] \neq Z_N$ for some $N$. [Hint: Use part (b) with $Z_1 = S_1$ and $Z_2 = S_2$ but $Z_3 \neq S_3$]

Problem 2. Consider a (homogenous) Markov-chain $(X_n)_{n \geq 0}$ on a finite state-space $S$ with transition matrix $P$. A function $f : S \to \mathbb{R}$ is considered as a column vector so that $Pf$ makes sense as matrix multiplication. Let $\mathcal{F}_n = \sigma(X_k : 0 \leq k \leq n)$.

(a) Check that $[Pf](X_n) = \mathbb{E}f(X_{n+1}) | \mathcal{F}_n$.

(b) Fix $f : S \to \mathbb{R}$ define

$$M_n = f(X_n) - f(X_0) - \sum_{k=0}^{n-1} [(P - I)f](X_k).$$

Show that $(M_n)_{n \geq 0}$ is a martingale.

(c) A function $f : S \to \mathbb{R}$ is called subharmonic if $f(x) \leq [Pf](x)$ for all $x$. Show that $(f(X_n))_{n \geq 0}$ a submartingale if $f$ is subharmonic. (This explains the ‘sub’ in the definition of submartingale.)

Problem 3.

(a) Given a sigma-algebra $\mathcal{G}$, show that $A \in \mathcal{G}$ if and only if $1_A$ is $\mathcal{G}$-measurable.

(b) Let $\tau$ be a stopping time for the filtration $(\mathcal{F}_n)_{n \geq 0}$ Show that $1_{\{\tau \geq n+1\}}$ is $\mathcal{F}_n$-measurable for all $n \geq 0$. 
(c) Let \( M = (M_n)_{n \geq 0} \) be a submartingale and \( \tau \) a stopping time. Show that the stopped submartingale \( M^\tau \) defined as

\[
M_n^\tau = M_{\tau \wedge n}
\]

is still a submartingale.

**Problem 4.** Let \( X_1, X_2, \ldots \) be i.i.d. random variables with \( \mathbb{E}(X_1) = \mu, \text{Var}(X_1) = \sigma^2 \) and moment generating function \( \phi(\theta) = \mathbb{E}[e^{\theta X_1}] \), where \( \phi \) is assumed finite valued. Assuming \( F_n = \sigma(X_1, \ldots, X_n) \), show that the following are martingales

(a) \( M_n = S_n^2 - \sigma^2 n \) if and only if \( \mu = 0 \).

(b) \( N_n = e^{\theta S_n} \phi(\theta)^{-n} \)

where \( S_n = X_1 + \ldots + X_n \).

**Problem 5.** Fix \( s \in \mathbb{Z} \), and suppose that \( X_1, X_2, \ldots \) are i.i.d. random variables with values in \( \{-1, 1\} \) so that \( \mathbb{P}(X_1 = 1) = p = 1 - \mathbb{P}(X_1 = -1) \) for some fixed \( p \in (0, 1) \). Let \( S \) be the process defined by \( S_0 = s \) and \( S_n = S_{n-1} + X_n \), i.e. a simple random walk started at \( S_0 = s \).

(a) Show that the processes \( M \) and \( N \) defined by

\[
M_n = \left( \frac{1-p}{p} \right)^{S_n} \quad \text{and} \quad N_n = S_n + n(1-2p)
\]

are martingales with respect to the filtration given by \( F_n = \sigma(X_1, \ldots, X_n) \).

Now assume \( p = 1/2 \), so that \( S \) is a simple symmetric random walk.

(b) Suppose \( S_0 = 1 \).

(i) Show that \( \tau = \inf\{n \geq 0, S_n = 0\} \) is a stopping time, possibly taking the value \( \infty \).

(ii) Apply the martingale convergence theorem to see that the stopped martingale \( S^\tau \) converges almost surely (to what?). Conclude that \( \tau < \infty \) a.s.

(iii) Show that the martingale \( S^\tau \) does not converge in \( L^1 \), i.e. \( \mathbb{E}(|S_n^\tau - S_{\infty}^\tau|) \) does not tend to 0 as \( n \to \infty \).

(c) Now let \( S \) be the simple symmetric random walk started at \( S_0 = 0 \).

(i) Fix integers \( a, b \geq 0 \) and let \( \tau = \inf\{n \geq 0, S_n = -a \text{ or } S_n = b\} \). Check that \( \tau \) is a stopping time. Why is \( \tau < \infty \) almost surely?

(ii) Use the optional stopping theorem to compute the probability that \( S \) hits \(-a\) before \( b \). Compute \( \mathbb{E}(\tau) \). [Hint: Show that \( S_n^2 - n \) defines a martingale, and apply the optional stopping theorem to it.]
Problem 6. At time 1 an urn contains a white and a red ball. Take out a ball at random and replace it by two balls of the same colour; this gives the new content of the urn at time 2. Keep iterating this procedure.

Let $Y_n$ be the number of white balls in the urn at time $n$, and let $X_n = \frac{Y_n}{n+1}$. Show that $X_n$ is a.s. convergent to a random variable $U$. Compute the mean of $U$. Can you compute the variance of $U$? [Hint: Consider the process $\frac{Y_n(Y_n+1)}{(n+1)(n+2)}$.]

Problem 7. Consider a single-period trinomial model, with two assets, a riskless bond and a risky stock. Suppose that initially both are worth $S_0^0 = S_0^1 = 1$. The riskless rate is $r$ so $S_0^1 = 1+r$. The risky asset at time 1 will be worth $a$ if the period was bad, $b$ if the period was indifferent, and $c$ if the period was good, $a < b < c$, and these are the only possibilities. We assume $a < 1+r < c$.

(a) Find all the risk-neutral measures for this model.

(b) For simplicity only, assume $r = 0$. Characterise all contingent claims with payout $Y = f(S_1^1)$ at time 1 that can be replicated, that is for which there exists $\pi \in \mathbb{R}^2$ such that

$$Y = \pi \cdot S_1^1.$$ 

Determine the price of this contingent claim at time 0. Compute the expectation of $Y$ with respect to any equivalent martingale measure. Conclusion?

(c) How would your analysis extend to a single-period model with $d + 1$ assets?

Problem 8. Consider a one-period binomial model with a stock and a riskless asset, that is $S^0$ and $S^1$ are defined as in Problem 7, but at time 1 the risky asset $S^1$ takes values $a$ and $c$ only. A utility-maximising investor has initial wealth $w_0 > 0$ and utility $U(x) = \sqrt{x}$. Find the agent’s optimal investment in the risky stock, and verify it has the same sign as $E[S_1^1] - (1+r)S_0$, where $r$ is the riskless interest rate, and $S_t$ is the price of the stock at time $t$.

Problem 9. Consider a single-period model with a risky asset $S^1$ having initial price $S_0^1$. At time 1 its value $S_1^1$ is a random variable on $(\Omega, \mathcal{F}, \mathbb{P})$ of the form

$$S_1^1 = \exp(\sigma Z + m), \quad m \in \mathbb{R}, \sigma > 0,$$

where $Z \sim N(0,1)$. ($S_1^1$ is then also said to be log-normal distributed). For simplicity assume that there is a riskless asset $S^0$ with $S_0^0 = S_1^0 = 1$ (so $r = 0$). Find a risk-neutral measure $\mathbb{Q}$ for this model. [Hint: Consider a density of the form $d\mathbb{Q} = d\mathbb{P}^* = \exp(\tilde{\sigma} Z + \tilde{m})$ and find suitable $\tilde{m}$ and $\tilde{\sigma}$.]