STOCHASTIC FINANCIAL MODELS: Examples 3 (of 4)

1. An investor with wealth $w_0$ at time 0 wishes to invest it in such a way as to maximise $E U(w_N/w_0)$, where $w_N$ is the wealth at the start of day $N$, and $U(x) = x^{1-R}/(1 - R)$. Each day, he chooses the proportion $\theta$ of his wealth to invest in the single risky asset, so that his wealth at the start of day $n+1$ will be

$$w_{n+1} = w_n\{\theta X_n + (1 - \theta)(1 + r)\}$$

where the $X_i$ are independent identically-distributed positive random variables, and $r$ is the per-period riskless rate of interest. Find form of the optimal policy in each of the two situations

(i) $\theta$ unrestricted;

(ii) $0 \leq \theta \leq 1$.

Show that the solutions are the same if and only if

$$\frac{E(X_1^{1-R})}{E(X_0^{1-R})} \leq 1 + r \leq EX_0.$$

2. At the start of year $n$, an investor receives income $Y_n \geq 0$. He chooses to invest a proportion $\theta_n \in [0,1]$ of this in the stock market, and consumes the remainder. At the start of the next year, his income is

$$Y_{n+1} = Y_n + \theta_n Y_n X_n,$$

where $X_n > 0$ is the return on the stock market in year $n$. The $X_n$ are independent identically-distributed random variables, with mean $\mu$. The investor’s objective is to choose the $\theta_n$ so as to maximise

$$E\{\sum_{n=0}^{N-1} (1 - \theta_n)Y_n + Y_N\}.$$

How should he invest so as to achieve this?

3. A gambler has the chance to bet on a sequence of $N$ coin tosses. Let $\xi_n = 1$ if the $n$th toss is a Head, and let $\xi_n = -1$ otherwise. The $\xi_n$ are independent but not identically distributed; $P(\xi_n = 1) = p_n \geq 1/2$. If the gambler’s wealth just before the $n$th toss is $w_{n-1}$, he may stake any amount $x \in [0, w_{n-1}]$ on the toss of the coin; his wealth at time $n$ is therefore

$$w_n = w_{n-1} + x\xi_n.$$ 

Determine how the gambler should play so as to maximise his final expected utility $E \log(w_N)$. 

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4. An agent starts with initial capital $C$. At the beginning of day $n$ ($n = 1, \ldots, N$) he observes a new non-negative random variable $Y_n$, and then chooses $c_n$, the amount of his remaining capital to consume that day. The $Y_n$ are independent, $\mathbb{E}Y_n = \mu_n$. His objective is to maximise

$$
\mathbb{E}\left[ \sum_{n=1}^{N} Y_n \log(c_n) \right]
$$

subject to the constraint $\sum_{n=1}^{N} c_n = C$. Find his optimal consumption policy, and an expression for the maximal value of his objective.

5. If $(B_t)_{t \geq 0}$ is a Brownian motion, show that the following processes are martingales:
   (i) $B_t^2 - t$;
   (ii) $\exp(\theta B_t - \frac{1}{2} \theta^2 t)$ for any $\theta \in \mathbb{R}$;
   (iii) $B_t^3 - 3tB_t$;
   (iv) $\cosh(\theta B_t)e^{-\theta^2 t/2}$ for any $\theta \in \mathbb{R}$.

6. An agent holds a single share, whose value at time $t$ is $S_0 + \sigma B_t$. The firm issuing the share may go bankrupt at some exponentially-distributed random time $T$ with mean $1/\lambda$. The agent plans to sell the share at the first time $H_a$ that the price exceeds $a$; if this comes before $T$, then the value to the agent is $a \exp(-rH_a)$, otherwise the value is nothing. Find the agent’s optimal choice of $a$.

7. Using the martingale from the second part of Question 5, and the Optional Sampling Theorem, show that if $H_a \equiv \inf\{t : B_t > a\}$ is the first time that a Brownian motion exceeds level $a > 0$, then the Laplace transform of $H_a$ is given by

$$
\mathbb{E}\exp(-\lambda H_a) = \exp(-a\sqrt{2\lambda}).
$$

Confirm this by integrating the density of $H_a$ as derived from the Reflection Principle.

8. Suppose that $a < x < b$, and let $\tau \equiv \inf\{t : B_t \not\in [a,b]\}$. For $\theta > 0$, using an appropriate martingale and the Optional Sampling Theorem, prove that

$$
\mathbb{E}[\exp(-\frac{1}{2} \theta^2 \tau) | B_0 = x] = \frac{\cosh(\theta(x - (a + b)/2))}{\cosh(\theta(b - a)/2)}.
$$