STOCHASTIC FINANCIAL MODELS: Examples 2 (of 4)

1. Suppose that $X_1, X_2, \ldots$ are IID real random variables with $E(|X_1|) < \infty$. Let $S_n := X_1 + \ldots + X_n$.

(a) When is $(S_n)$ a martingale? Specify the filtration.

(b) Show that $\mathbb{E}[X_1|S_n] = \frac{S_n}{n}$.

(c) Compute $\mathbb{E}[S_n|X_1]$.

(d) Find an example of a process $(Z_n)_{n \in I}$ adapted to some filtration $(\mathcal{F}_n)_{n \in I}$ which satisfies $\mathbb{E}[Z_{n+1}|Z_n] = Z_n$ but $\mathbb{E}[Z_{n+1}|\mathcal{F}_n] \neq Z_n$.

(Hint: Take $X \sim N(0, 1)$, $I = \{1, 2, 3\}$, $Z_1 = X_1$, $Z_2 = X_1 + X_2$ and construct $Z_3 - Z_2$ independent of $Z_2$ but fully determined at times 1 and 2.)

2. Consider a (homogenous) Markov-chain $(X_n)_{n \in \mathbb{Z}^+}$ on a finite state-space $S$ with transition matrix $P$. A function $f : S \to \mathbb{R}$ is considered as a column vector so that $Pf$ makes sense as matrix multiplication.

(a) Let $\mathbb{E}^x$ indicate that $X_0 = x \in S$. Check that $[Pf](x) = \mathbb{E}^x f(X_1)$.

(b) Consider an arbitrary function $g : S \to \mathbb{R}$. Show that $g(X_n)$ is $\sigma(X_n)$-measurable. Set $\mathcal{F}_n = \sigma(X_k : 0 \leq k \leq n)$. Why is $g(X_n)$ is $\mathcal{F}_n$-measurable?

(c) Fix $f : S \to \mathbb{R}$ and define

$$M_n = f(X_n) - f(X_0) - \sum_{k=0}^{n-1} [(P - I) f](X_k).$$

Show that $(M_n, \mathcal{F}_n)$ is a martingale.

(d) Call a function $f : S \to \mathbb{R}$ sub-harmonic if $f(x) \leq [Pf](x)$ for all $x$. Show that $f(X_n)$ is a sub-martingale w.r.t. the filtration $(\mathcal{F}_n)$. (This explains the "sub" in the definition of sub-martingale.)

3. (a) Given a $\sigma$-algebra $\mathcal{F}$, show that $A \in \mathcal{F}$ if and only $1_A$ is $\mathcal{F}$-measurable.

(b) Let $\tau$ be a stopping time with respect to the filtration $(\mathcal{F}_t)_{t \in \mathbb{Z}^+}$. Show that $1_{\{\tau \geq t+1\}}$ is $\mathcal{F}_t$-measurable for all $t \in \mathbb{Z}^+$.

(c) Let $(M_t, \mathcal{F}_t)_{t \in \mathbb{Z}^+}$ be a martingale and $\tau$ a stopping time w.r.t. the filtration $(\mathcal{F}_t)_{t \in \mathbb{Z}^+}$. Show directly (without appealing to martingale transforms) that the stopped martingale

$$M_t^\tau := M_{t \wedge \tau}$$

is still a martingale w.r.t. the filtration $(\mathcal{F}_t)_{t \in \mathbb{Z}^+}$.
4. (a) Suppose that $X_1, X_2, \ldots$ are IID random variables with values in $\{-1, +1\}$ so that $\mathbb{P}(X_1 = 1) = p$, $\mathbb{P}(X_1 = -1) = 1 - p =: q$ for some fixed $p \in (0, 1)$. The process

$$S_0 = 0, S_t := X_1 + \ldots + X_t, \quad t \in \mathbb{N}.$$ 

is known as the $p/q$-simple random walk started at 0 $\in \mathbb{R}$. Show that the processes

$$M_t := (q/p)^{S_t}, \quad N_t := S_t - \mathbb{E}S_t$$

are martingales w.r.t. the filtration given by $\mathcal{F}_t := \sigma(X_1, \ldots, X_t)$.

(b) Now assume $p = q = 1/2$, this gives rise to the simple symmetric random walk $S$. Assume the random walk is started $S_0 = 1$. Show that $\tau = \inf \{t \in \mathbb{Z}^+ : S_t = 0\}$ is a stopping time, possibly of value $+\infty$.

(c) Apply the martingale convergence theorem to see that the stopped martingale $S^\tau$ converges almost surely (to what?). Conclude that $\tau < \infty$ a.s..

* (d) Show that the martingale $S^\tau$ does not converge in $L^1$.

(e) Now let $S$ be the simple symmetric random walk $S$ started at $S_0 = 0$. Let $a, b \in \mathbb{N}$, check that $\tau = \inf \{t \in \mathbb{Z}^+ : S_t = -a \text{ or } S_t = b\}$ is a stopping time. Why is $\tau < \infty$ almost surely?

(f) Use the optional stopping (sampling) theorem to compute the probability that $S$ hits $a$ before $b$. Find the expected time to hit $-a$ or $b$.

5. Let $X_1, X_2, \ldots$ be IID random variables with $\mathbb{E}X_1 = \mu$, $\mathbb{V}arX_1 = \sigma^2$, and $\varphi(\theta) = \mathbb{E}e^{\theta X_1}$, finite-valued for all $\theta$. Let $S_n = X_1 + \ldots + X_n$, $S_0 = 0$. Assuming that at time $n$ the values of $X_1, \ldots, X_n$ are known, show that the following are martingales:

(a) $\{S_k^2 - \sigma^2 k : k \geq 0\}$ if and only if $\mu = 0$;

(b) $\{e^{\theta S_k} \varphi(\theta)^{-k} : k \geq 0\}$.

6. At time 1 an urn contains a white and a red ball. Take out a ball at random and replace it by two balls of the same color; this gives the new content of the urn at time 2. Keep iterating this procedure.

Let $Y_n$ be the number of white balls in the urn at time $n$, and let $X_n = Y_n/(n + 1)$. Show that $X_n$ is a.s. convergent to a random variable $U$. Compute the mean of $U$. Can you compute the variance of $U$?

7. Prove that the existence of an EMM implies no-arbitrage in the discrete multi-period setting. (You may assume $S_t^0 \equiv 1$ for all $t$.)
8. Suppose that over two periods a stock price moves on a binomial tree

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   45
   /   \
 30   36
  /     \
15
  /     \
 12    16
  /      \
 10
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Determine for what values of the riskless rate $r$ there is no arbitrage. If $r = 1/4$, determine the equivalent martingale measure. With this value of $r$, find the time-zero price and replicating portfolio for a European put option with strike 15 and expiry 2.

9. Consider a single-period trinomial model, with two assets, a riskless bond $S^0$ and a risky share $S^1$. We have initially that both are worth 1:

$$S_0^0 = 1 = S_0^1,$$

and that $S_1^0 = 1 + r$. The risky asset at time 1 will be worth $a$ if the period was bad, $b$ if the period was indifferent, and $c$ if the period was good, $a < b < c$, and these are the only possibilities. We assume $a < 1 + r < c$.

(a) Find all the equivalent martingale measures for this model.

(b) For simplicity only, assume $r = 0$. Characterise all contingent claim with payoff $Y = (y_a, y_b, y_c)$ at time 1 that can be replicated, that is for which there exists $\theta \in \mathbb{R}^2$ such that

$$Y = \theta \cdot S_1$$

Determine the price of this contingent claim at time 0. Compute the expectation of $Y$ with respect to any equivalent martingale measure. Conclusion?

(c) How would your analysis extend to a single-period model with $n$ assets?

10. A utility-maximising investor in a one-period binomial model has initial wealth $w_0 > 0$ and utility $U(x) = \sqrt{x}$. If $S^0$ denotes the riskless asset (worth 1 at time 0 and $1 + r$ at time 1), and $S^1$ denotes the risky asset, find the agent’s optimal investment in the stock, and verify that this has the same sign as $\mathbb{E}[S_1^1 - (1 + r)S_0^1]$. 