DIFFERENTIAL GEOMETRY EXAMPLES 3

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Comments/corrections are welcome, and may be e-mailed to me at `pmhw@dpmms.cam.ac.uk`.

1. Let $\alpha : I \to S$ be a geodesic. Show that if α is a plane curve and $\ddot{\alpha}(t) \neq 0$ for some $t \in I$, then $\dot{\alpha}(t)$ is an eigenvector of the differential of the Gauss map at $\alpha(t)$. [Hint: without loss of generality suppose that α is parametrized by arc-length and observe that the normal to α and the normal to the surface have to be collinear around t.]

2. Show that if all geodesics of a connected surface are plane curves, then the surface is contained in a plane or a sphere [Hint: use the previous problem and Problem 12 of Example sheet 2].

3. Let $f : S_1 \to S_2$ be an isometry between two surfaces.

(i) Let $\alpha : I \to S_1$ be a curve and V a vector field along α. Let $\gamma := f \circ \alpha$, and $W(t) := df_{\alpha(t)}(V(t))$ the corresponding vector field along γ. Show that $DW/dt = df_{\alpha(t)}(DV/dt)$, and hence that V parallel along α implies that W is parallel along γ.

(ii) Deduce that f maps geodesics to geodesics.

4. Show that the equations for geodesics on a smooth surface may be written locally in terms of coordinates $(u(t), v(t))$ as

$$\frac{d}{dt}(E\dot{u} + F\dot{v}) = \frac{1}{2}(E_u\dot{u}^2 + 2F_u\dot{u}\dot{v} + G_u\dot{v}^2)$$
$$\frac{d}{dt}(F\dot{u} + G\dot{v}) = \frac{1}{2}(E_v\dot{u}^2 + 2F_v\dot{u}\dot{v} + G_v\dot{v}^2).$$

5. Consider the surface of revolution from Problem 9, Example sheet 2.

(i) Write down the differential equations of the geodesics;

(ii) Establish *Clairaut's relation*: $f^2\dot{u}$ is constant along geodesics. Show that if θ is the angle that a geodesic makes with a parallel and r is the radius of the parallel at the intersection point, then Clairaut's relation says that $r\cos\theta$ is constant along geodesics.

(iii) Show that meridians are geodesics; when is a parallel a geodesic?

6. Show that there are no compact minimal surfaces in $\mathbb{R}^3$.

7. The existence of isothermal coordinates is a hard theorem. However for the case of minimal surfaces without planar points it is possible to give an easy proof along the following lines.

(i) Let S be a regular surface without umbilical points. Prove that S is a minimal surface if and only if the Gauss map $N : S \to S^2$ satisfies

$$\langle dN_p(v_1), dN_p(v_2)\rangle = \lambda(p)\langle v_1, v_2\rangle$$

for all $p \in S$ and all $v_1, v_2 \in T_pS$, where $\lambda(p) \neq 0$ is a number which depends only on p.

(ii) By considering stereographic projection and (i) show that isothermal coordinates exist around a non planar point in a minimal surface.

For the next five questions we consider the Weierstrass representation of a minimal surface determined by functions f and g on a simply connected domain $D \subseteq \mathbb{C}$ as we saw in lectures.

8. Show that if ϕ is the parametrization defined by the Weierstrass representation, then ϕ is an immersion if and only f vanishes only at the poles of g and the order of its zero at such a point is *exactly* twice the order of the pole of g.

9. Find D, f and g representing the catenoid and the helicoid.

10. Show that the Gaussian curvature of the minimal surface determined by the Weierstrass representation is given by

$$K = -\left(\frac{4\,|g'|}{|f|\,(1+|g|^2)^2}\right)^2.$$

Show that either $K \equiv 0$ or its zeros are isolated. [There is a way of doing this problem almost without calculations. Think about the relation between g and the Gauss map and the fact that stereographic projection is conformal.]

11. The Weierstrass representation is not unique: if $\phi_{(f,g)} : D \to \mathbb{R}^3$ is the associated parametrization and $\alpha : W \to D$ is a bijective holomorphic map, then $\phi_{(f,g)} \circ \alpha$ is another representation of the same minimal surface and it must have the same form with different f and g (which should be specified). By choosing $\alpha(z) = g^{-1}(z)$, show that, locally around regular points of g at which g' is non-zero, we can assume that our pair (f, g) is of the form (F, id), for some local holomorphic function F. We denote such a representation by ϕ_F.

12. Show that the minimal surfaces given by $\phi_{e^{-i\theta}F}$ for θ real are all locally isometric. With an appropriate choice of F, show that the catenoid and the helicoid are locally isometric. Show however that the catenoid comes from embedding $\mathbf{C}^*$ into $\mathbf{R}^3$, whilst the helicoid comes from embedding $\mathbf{C}$.

13*. The intrinsic distance of a smooth embedded surface $S \subset \mathbb{R}^3$ is defined as follows. Given p and q in S let $d(p,q) = \inf_{\alpha\in\Omega(p,q)} \ell(\alpha)$. Show that d is a metric, which is compatible with the topology of S. If S is complete (and without boundary) the Hopf-Rinow theorem asserts that given two points p and q there exists a geodesic γ joining the points such that $d(p,q) = \ell(\gamma)$ and geodesics are defined for all $t \in \mathbb{R}$.

(i) Show that if $f : S_1 \to S_2$ is an isometry, then $d_2(f(p), f(q)) = d_1(p,q)$ for all p and q in S_1.

(ii) A geodesic $\gamma : [0,\infty) \to S$ is called a *ray leaving from* p if it realizes the distance between $\gamma(0)$ and $\gamma(s)$ for all $s \in [0,\infty)$. Let p be a point in a complete, noncompact surface S. Prove that S contains a ray leaving from p. [You may assume that geodesics vary smoothly (hence continuously) with their initial conditions.]

14*. Show that any geodesic of the paraboloid of revolution $z = x^2 + y^2$ which is not a meridian intersects itself an infinite number of times [Hint: use Clairaut's relation. You may assume that no geodesic of a surface of revolution can be asymptotic to a parallel which is not itself a geodesic. You will need to show that for a geodesic which is not a meridian, $u(t)$ does not approach some u_0 as $t \to \infty$.]