DIFFERENTIAL GEOMETRY EXAMPLES 4

P.M.H. Wilson, Lent 2015 pmhw@dpmms.cam.ac.uk

1. Consider the standard (Euclidean) inner product on the space $M(n)$ of real $n \times n$ matrices, namely $\langle L, K \rangle = \mathrm{Tr}(LK^t)$ where K^t denotes the transpose matrix to K, and the induced metric on the tangent spaces to $X = \mathrm{O}(n) \subset M(n)$.

For $A \in T_I X$, consider the curve $\alpha : \mathbf{R} \to M(n)$ given by $\alpha(t) = \exp(tA)$, as defined in lectures. Prove that α is a curve on X and that it is geodesic, that is $\alpha''(t) = A^2\alpha(t)$ is orthogonal to $T_{\alpha(t)}X$ for all $t \in \mathbf{R}$.

2. Using geodesic polar coordinates, show that given $p \in S$ we can express the Gaussian curvature as

$$K(p) = \lim_{r \to 0} \frac{3(2\pi r - L)}{\pi r^3},$$

where L is the length of the geodesic circle of radius r [Hint: Taylor expansion for $\sqrt{G}$; you may assume that the remainder term is well-behaved in θ].

3. Find the geodesic curvature of a parallel of latitude on the 2-sphere.

4. Prove that on a surface of constant Gaussian curvature, the geodesic circles have constant geodesic curvature, namely $G_\rho/2G$ where in geodesic polar coordinates the first fundamental form is $d\rho^2 + G(r, \theta)d\theta^2$. Suppose that on a surface S, we have a point P with the property that locally around P the Gaussian curvature is constant along each geodesic circle; show that the geodesic curvature is also constant along these geodesic circles.

5. Let S be a connected surface and $f, g : S \to S$ two isometries. Assume that there exists $p \in S$, such that $f(p) = g(p)$ and $df_p = dg_p$. Show that $f(q) = g(q)$ for all $q \in S$.

6. (Geodesics are local minimizers of length.) Let p be a point on a surface S. Show that there exists an open set V containing p such that if $\gamma : [0, 1] \to V$ is a geodesic with $\gamma(0) = p$ and $\gamma(1) = q$ and $\alpha : [0, 1] \to S$ is a regular curve joining p to q, then

$$\ell(\gamma) \leq \ell(\alpha)$$

with equality if and only if α is a monotonic reparametrization of γ.

7. Let P be a point on an embedded surface $S \subset \mathbf{R}^3$; consider the orthogonal parametrization $\phi : (-\epsilon, \epsilon)^2 \to V \subset S$ of a neighbourhood of P as constructed in lectures, where the curve $\phi(0, v)$ is a geodesic of unit speed, and for any $v_0 \in (-\epsilon, \epsilon)$ the curve $\phi(u, v_0)$ is a geodesic of unit speed. We showed that the first fundamental form was then $du^2 + G(u, v)dv^2$ for some smooth function G. Prove that $G(u, v) = 1$ for all u, v if and only if the curves $\phi(u_0, v)$ are geodesics for all $u_0 \in (-\epsilon, \epsilon)$.

8. Let S be a compact connected orientable surface which is not diffeomorphic to a sphere. Prove that there are points on S where the Gaussian curvature is positive, negative, and zero.

9. Let S be a compact oriented surface with positive Gaussian curvature and let $N : S \to S^2$ be the Gauss map. Let γ be a simple closed geodesic in S, and let A and B be the regions which have γ as a common boundary. Show that $N(A)$ and $N(B)$ have the same area.

10. Let S be an orientable surface with Gaussian curvature $K \leq 0$. Show that two geodesics γ_1 and γ_2 which start from a point $p \in S$ will not meet again at a point q in such a way that the traces (i.e. images) of γ_1 and γ_2 form the boundary of a domain homeomorphic to a disk.

11. Let S be a surface homeomorphic to a cylinder and with negative Gaussian curvature. Show that S has at most one simple closed geodesic.

12. Let $\phi : U \to S$ be an orthogonal parametrization around a point p. Let $\alpha : [0, \ell] \to \phi(U)$ be a smooth simple closed curve parametrized by arc-length enclosing a domain R. Fix a unit vector $w_0 \in T_{\alpha(0)}S$ and consider $W(t)$ the parallel transport of w_0 along α. Let $\psi(t)$ be a differentiable determination of the angle from ϕ_u to $W(t)$. Show that

$$\psi(\ell) - \psi(0) = \int_R K\, dA.$$

Let S be a connected surface. Use the above to show that if the parallel transport between any two points does not depend on the curve joining the points, then the Gaussian curvature of S is zero.

13 If $a > 0$, calculate the curvature and torsion of the smooth curve given by

$$\alpha(s) = (a\cos(s/c), a\sin(s/c), bs/c) \quad \text{where } c = \sqrt{a^2 + b^2}.$$

Suppose now that $\alpha : [0, 2\pi] \to \mathbf{R}^3$ is a smooth simple closed curve parametrized by arc-length with curvature everywhere positive. If both k and τ are constant, show that $k = 1$ and $\tau = 0$. If k is constant and τ is not identically zero, show that $k > 1$. If α is knotted and τ is constant, show that $k(s) > 2$ for some $s \in [0, 2\pi]$.

The remaining two questions complete a circle of ideas in the course. They are more ambitious than the previous ones and their content is certainly not examinable, and so they should be regarded as optional.

14. (The Poincaré-Hopf theorem.) Let S be an oriented surface and $V : S \to \mathbb{R}^3$ a smooth vector field, that is, $V(p) \in T_pS$ for all $p \in S$. We say that p is *singular* if $V(p) = 0$. A singular point p is *isolated* if there exists a neighbourhood of p in which V has no other zeros. The singular point p is *non-degenerate* if $dV_p : T_pS \to T_pS$ is a linear isomorphism (can you see why dV_p takes values in T_pS?). Show that if a singular point is non-degenerate, then it is isolated.

To each isolated singular point p we associate an integer called the *index* of the vector field at p as follows. Let $\phi : U \to S$ be an orthogonal parametrization around p compatible with the orientation. Let $\alpha : [0, l] \to \phi(U)$ be a regular piecewise smooth simple closed curve so that p is the only zero of V in the domain enclosed by α. Let $\varphi(t)$ be some differentiable determination of the angle from ϕ_u to $V(t) := V \circ \alpha(t)$. Since α is closed, there is an integer I (the index) defined by

$$2\pi\, I := \varphi(l) - \varphi(0).$$

(i) Show that I is independent of the choice of parametrization (Hint: use Problem 12). One can also show that I is independent of the choice of curve α, but this is a little harder. Also one can prove that if p is non-degenerate, then $I = 1$ if dV_p preserves orientation and $I = -1$ if dV_p reverses orientation.

(ii) Draw some pictures of vector fields in $\mathbb{R}^2$ with an isolated singularity at the origin. Compute their indices.

(iii) Suppose now that S is compact and that V is a smooth vector field with isolated singularities. Consider a triangulation of S such that

- every triangle is contained in the image of some orthogonal parametrization;
- every triangle contains at most one singular point;
- the boundaries of the triangles contain no singular points and are positively oriented.

Show that

$$\sum_i I_i = \frac{1}{2\pi}\int_S K\, dA = \chi(S).$$

Thus, you have proved that the sum of the indices of a smooth vector field with isolated singularities on a compact surface is equal to the Euler characteristic (Poincaré-Hopf theorem). Conclude that a surface homeomorphic to S^2 cannot be combed.

Finally, suppose $f : S \to \mathbb{R}$ is a Morse function and consider the vector field given by the gradient of f, i.e., $\nabla f(p)$ is uniquely determined by $\langle \nabla f(p), v\rangle = df_p(v)$ for all $v \in T_pS$. Use the Poincaré-Hopf theorem to show that $\chi(S)$ is the number of local maxima and minima minus the number of saddle points. Use this to find the Euler characteristic of a surface of genus two.

15. (The degree of the Gauss map.) Let S be a compact oriented surface and let $N : S \to S^2$ be the Gauss map. Consider $y \in S^2$ a regular value. Rather than counting their preimages modulo 2 as we did in the first lectures, we will count them with sign. Let $N^{-1}(y) = \{p_1, \dots, p_n\}$. Let $\varepsilon(p_i)$ be $+1$ if dN_{p_i} preserves orientation ($K(p_i) > 0$), and -1 if dN_{p_i} reverses orientation ($K(p_i) < 0$). Now let

$$\deg(N) := \sum_i \varepsilon(p_i).$$

As in the case of the degree mod 2, it can be shown that the sum on the right hand side is independent of the regular value and $\deg(N)$ turns out to be an invariant of the homotopy class of N.

Now, choose $y \in S^2$ such that y and $-y$ are regular values of N. Why can we do so? Let V be the vector field on S given by

$$V(p) := \langle y, N(p) \rangle N(p) - y.$$

(i) Show that the index of V at a zero p_i is $+1$ if dN_{p_i} preserves orientation and -1 if dN_{p_i} reverses orientation.

(ii) Show that the sum of the indices of V equals twice the degree of N.

(iii) Show that $\deg(N) = \chi(S)/2$.