Part IID DIFFERENTIAL GEOMETRY (Lent 2011): Example Sheet 4

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1. Using geodesic polar coordinates, show that given $p \in S$ we can express the Gaussian curvature as

$$K(p) = \lim_{r \to 0} \frac{3(2\pi r - L)}{\pi r^3},$$

where L is the length of the geodesic circle of radius r [Hint: Taylor expansion].

2. Let $\alpha : I \to S$ be a curve parametrized by arc-length on an oriented surface S. Show that $k^2 = k_g^2 + k_n^2$, where k is the curvature of α, k_g is its geodesic curvature, and k_n is the normal curvature.

3. Find the geodesic curvature of a parallel of latitude on the 2-sphere.

4. Prove that on a surface of constant Gaussian curvature, the geodesic circles have constant geodesic curvature.

5. Let S be a connected surface and $f, g : S \to S$ two isometries. Assume that there exists $p \in S$, such that $f(p) = g(p)$ and $df_p = dg_p$. Show that $f(q) = g(q)$ for all $q \in S$.

6. (Geodesics are local minimizers of length.) Let p be a point on a surface S. Show that there exists an open set V containing p such that if $\gamma : [0,1] \to V$ is a geodesic with $\gamma(0) = p$ and $\gamma(1) = q$ and $\alpha : [0,1] \to S$ is a regular curve joining p to q, then

$$\ell(\gamma) \leq \ell(\alpha)$$

with equality if and only if α is a reparametrization of γ.

7. Show that in a system of normal coordinates centered at p (i.e. cartesian coordinates (x, y) in T_pS and parametrization $(x, y) \mapsto \exp_p(xe_1 + ye_2)$), all the Christoffel symbols are zero at p.

8. Let S be a compact connected orientable surface which is not diffeomorphic to a sphere. Prove that there are points on S where the Gaussian curvature is positive, negative and zero.

9. Let S be a compact oriented surface with positive Gaussian curvature and let $N : S \to S^2$ be the Gauss map. Let γ be a simple closed geodesic in S, and let A and B be the regions which have γ as a common boundary. Show that $N(A)$ and $N(B)$ have the same area.

10. Let S be an orientable surface with Gaussian curvature $K \leq 0$. Show that two geodesics γ_1 and γ_2 which start from a point $p \in S$ will not meet again at a point q in such a way that the traces (i.e. images) of γ_1 and γ_2 form the boundary of a domain homeomorphic to a disk.

11. Let S be a surface homeomorphic to a cylinder and with negative Gaussian curvature. Show that S has at most one simple closed geodesic.

12. Let $\varphi : U \to S$ be an orthogonal parametrization around a point p. Let $\alpha : [0, \ell] \to \varphi(U)$ be a simple closed curve parametrized by arc-length enclosing a domain R. Fix a unit vector $w_0 \in T_{\alpha(0)}S$ and consider $W(t)$ the parallel transport of w_0 along α. Let $\psi(t)$ be a differentiable determination of the angle from φ_u to $W(t)$. Show that

$$\psi(\ell) - \psi(0) = \int_R K dA.$$

Let S be a connected surface. Use the above to show that if the parallel transport between any two points does not depend on the curve joining the points, then the Gaussian curvature of S is zero.

The remaining two questions are more ambitious than the previous ones and their content is certainly not examinable. Nevertheless I hope that you will enjoy thinking about some of them.

13. (The Poincaré–Hopf theorem.) Let S be an oriented surface and $V : S \to \mathbb{R}^3$ a smooth vector field, that is, $V(p) \in T_pS$ for all $p \in S$. We say that p is *singular* if $V(p) = 0$. A singular point p is *isolated* if there exists a neighbourhood of p in which V has no other zeros. The singular point p is *non-degenerate* if $dV_p : T_pS \to T_pS$ is a linear isomorphism (can you see why dV_p takes values in T_pS?). Show that if a singular point is non-degenerate, then it is isolated.

To each isolated singular point p we associate an integer called the *index* of the vector field at p as follows. Let $\varphi : U \to S$ be an orthogonal parametrization around p compatible with the orientation. Let $\alpha : [0, l] \to \varphi(U)$ be a regular piecewise smooth simple closed curve, so that p is the only zero of V in the domain enclosed by α. Let $\psi(t)$ be some differentiable determination of the angle from φ_u to $V(t) := V \circ \alpha(t)$. Since α is closed, there is an integer I (the index) defined by

$$2\pi I := \psi(l) - \psi(0).$$

(i) Show that I is independent of the choice of parametrization (Hint: use Problem 12). One can also show that I is independent of the choice of curve α, but this is a little harder. Also one can prove that if p is non-degenerate, then $I = 1$ if dV_p preserves orientation and $I = -1$ if dV_p reverses orientation.

(ii) Draw some pictures of vector fields in $\mathbb{R}^2$ with an isolated singularity at the origin. Compute their indices.

(iii) Suppose now that S is compact and that V is a smooth vector field with isolated singularities. Consider a triangulation of S such that

- every triangle is contained in the image of some orthogonal parametrization;
- every triangle contains at most one singular point;
- the boundaries of the triangles contain no singular points and are positively oriented.

Show that

$$\sum_i I_i = \frac{1}{2\pi} \int_S K dA = \chi(S).$$

Thus, you have proved that the sum of the indices of a smooth vector field with isolated singularities on a compact surface is equal to the Euler characteristic (Poincaré–Hopf theorem). Conclude that a surface homeomorphic to S^2 cannot be 'combed'.

Finally, suppose $f : S \to \mathbb{R}$ is a Morse function and consider the vector field given by the gradient of f, i.e. $\nabla f(p)$ is uniquely determined by $\langle \nabla f(p), v \rangle = df_p(v)$ for all $v \in T_pS$. Use the Poincaré–Hopf theorem to show that $\chi(S)$ is the number of local maxima and minima minus the number of saddle points. Use this to determine the Euler characteristic of a surface of genus two.

14. (The degree of the Gauss map.) Let S be a compact oriented surface and let $N : S \to S^2$ be the Gauss map. Consider $y \in S^2$ a regular value. Rather than counting their preimages modulo 2 as we did in the first lectures, we will count them *with sign.* Let $N^{-1}(y) = \{p_1, \ldots, p_n\}$. Let $\varepsilon(p_i)$ be $+1$ if dN_{p_i} preserves orientation ($K(p_i) > 0$), and -1 if dN_{p_i} reverses orientation ($K(p_i) < 0$). Now let

$$\deg(N) := \textstyle\sum_i \varepsilon(p_i).$$

As in the case of the degree mod 2, it can be shown that the sum on the right hand side is independent of the regular value and $\deg(N)$ turns out to be an invariant of the homotopy class of N.

Now, choose $y \in S^2$ such that y and $-y$ are regular values of N. Why can we do so? Let V be the vector field on S given by

$$V(p) := \langle y, N(p) \rangle N(p) - y.$$

(i) Show that the index of V at a zero p_i is $+1$ if dN_{p_i} preserves orientation and -1 if dN_{p_i} reverses orientation.

(ii) Show that the sum of the indices of V equals twice the degree of N.

(iii) Show that $\deg(N) = \chi(S)/2$.