

Part IID DIFFERENTIAL GEOMETRY (Lent 2011): Example Sheet 3

Comments, corrections are welcome at any time.

a.g.kovalev@dpmms.cam.ac.uk.

1. Let $\alpha : I \rightarrow S$ be a geodesic. Show that if α is a plane curve and $\ddot{\alpha}(t) \neq 0$ for some $t \in I$, then $\dot{\alpha}(t)$ is an eigenvector of the differential of the Gauss map at $\alpha(t)$.

[Hint: without loss of generality suppose that α is parametrized by arc-length and observe that the normal to α and the normal to the surface have to be colinear around t .]

2. Show that if all geodesics of a connected surface are plane curves, then the surface is contained in a plane or a sphere.

[Hint: use the previous problem and Problem 15 of Example sheet 2].

3. Let $f : S_1 \rightarrow S_2$ be an isometry between two surfaces.

(i) Let $\alpha : I \rightarrow S_1$ be a curve and V a parallel vector field along α . Show that $df_{\alpha(t)}(V(t))$ is a parallel vector field along $f \circ \alpha$.

(ii) Show that f preserves geodesics.

4. Consider the surface of revolution from Problem 10, Example sheet 2.

(i) Find the differential equations of the geodesics;

(ii) Establish *Clairaut's relation*: $f^2 \dot{u}$ is constant along geodesics. Show that if θ is the angle that a geodesic makes with a parallel and r is the radius of the parallel at the intersection point, then Clairaut's relation says that $r \cos \theta$ is constant along geodesics.

(iii) Show that meridians are geodesics; when is a parallel a geodesic?

5. Show that there are no compact minimal surfaces in $\mathbb{R}^3$.

6. Let S_1 and S_2 be surfaces with Gaussian curvatures K_{S_1} and K_{S_2} , respectively, and $f : S_1 \rightarrow S_2$ a diffeomorphism.

(i) Suppose that $K_{S_2}(f(x)) = K_{S_1}(x)$. Must f be an isometry?

(ii) Suppose instead that f maps geodesics of S_1 to geodesics of S_2 . Must f be an isometry?

7. The existence of isothermal coordinates is a hard theorem. However for the case of minimal surfaces without planar points it is possible to give an easy proof along the following lines.

(i) Let S be a regular surface without umbilical points. Prove that S is a minimal surface if and only if the Gauss map $N : S \rightarrow S^2$ satisfies

$$\langle dN_p(v_1), dN_p(v_2) \rangle = \lambda(p) \langle v_1, v_2 \rangle$$

for all $p \in S$ and all $v_1, v_2 \in T_p S$, where $\lambda(p) \neq 0$ is a number which depends only on p .

(ii) By considering stereographic projection and (i) show that isothermal coordinates exist around a non planar point in a minimal surface.

8. The intrinsic distance of a surface S is defined as follows. Given p and q in S let $d(p, q) = \inf_{\alpha \in \Omega(p, q)} \ell(\alpha)$. It can be shown that d is a distance (can you see which property requires some care?) which is compatible with the topology of S . If S is complete (and without boundary) the Hopf–Rinow theorem asserts that given two points p and q there exists a geodesic γ joining the points such that $d(p, q) = \ell(\gamma)$ and geodesics are defined for all $t \in \mathbb{R}$.

(i) Show that if $f : S_1 \rightarrow S_2$ is an isometry, then $d_2(f(p), f(q)) = d_1(p, q)$ for all p and q in S_1 .

(ii) A geodesic $\gamma : [0, \infty) \rightarrow S$ is called a *ray leaving from p* if it realizes the distance between $\gamma(0)$ and $\gamma(s)$ for all $s \in [0, \infty)$. Let p be a point in a complete, noncompact surface S . Prove that S contains a ray leaving from p . [You may assume that geodesics vary smoothly (hence continuously) with their initial conditions.]

For the next five questions we consider the Weierstrass representation of a minimal surface determined by functions f and g on a simply connected domain $D \subseteq \mathbb{C}$ as we saw in lectures.

9. Show that if φ is the parametrization defined by the Weierstrass representation, then φ is an immersion if and only if f vanishes only at the poles of g and the order of its zero at such a point is *exactly* twice the order of the pole of g .

10. Find D , f and g representing the catenoid $\varphi = (a \cosh v \cos u, a \cosh v \sin u, av)$ and the helicoid $\varphi = (a \sinh v \cos u, a \sinh v \sin u, au)$.

11. Show that the Gaussian curvature of the minimal surface determined by the Weierstrass representation is given by

$$K = - \left(\frac{4|g'|}{|f|(1 + |g|^2)^2} \right)^2$$

Show that either $K \equiv 0$ or its zeros are isolated. [There is a way of doing this problem almost without calculations. Think about the relation between g and the Gauss map and the fact that stereographic projection is conformal.]

12. The Weierstrass representation is not unique: if $\varphi_{(f,g)} : D \rightarrow \mathbb{R}^3$ is the associated parametrization and $\alpha : W \rightarrow D$ is a bijective holomorphic map, then $\varphi_{(f,g)} \circ \alpha$ is another representation of the same minimal surface and it must have the same form with different f and g . Show that by choosing $\alpha(z) = g^{-1}(z)$ we can assume that our pair (f, g) is of the form (F, id) (g can be inverted near a point which is not a pole and for which g' is not zero). We denote such a representation by φ_F .

13. Show that the minimal surfaces given by $\varphi_{e^{-i\theta}F}$ for θ real are all locally isometric. With an appropriate choice of F , show that the catenoid and the helicoid are locally isometric.

14. Show that any geodesic of the paraboloid of revolution $z = x^2 + y^2$ which is not a meridian intersects itself an infinite number of times.

[Hint: use Clairaut's relation. You may assume that no geodesic of a surface of revolution can be asymptotic to a parallel which is not itself a geodesic. You will need to show that for a geodesic which is not a meridian, $u(t)$ does not approach some u_0 as $t \rightarrow \infty$.]

15. Suppose that on a connected surface S there is a point p such that all geodesics through p are closed, i.e. all geodesics through p induce smooth maps $S^1 \rightarrow S$. Show that S is compact.