Part IID DIFFERENTIAL GEOMETRY (Lent 2011): Example Sheet 1

Comments, corrections are welcome at any time. `a.g.kovalev@dpmms.cam.ac.uk`.

1. If X and Y are manifolds, show that $X \times Y$ is a manifold of dimension $\dim X \times Y = \dim X + \dim Y$.

2. Let X be a submanifold of Y and suppose that X and Y have the same dimension. Show that X is an open subset of Y.

3. Let B_r denote the open ball $\{x \in \mathbb{R}^k : |x| < r\}$. Show that the map

$$x \in B_r \to \frac{rx}{\sqrt{r^2 - |x|^2}} \in \mathbb{R}^k$$

is a diffeomorphism. (This implies that local parametrizations can always be chosen with all of $\mathbb{R}^k$ as domain.)

4. (i) Is the union of two coordinate axes in $\mathbb{R}^2$ a manifold?

(ii) Prove that the hyperboloid in $\mathbb{R}^3$ given by $x^2 + y^2 - z^2 = a$ is a manifold for $a > 0$. What happens for $a = 0$? Find the tangent space at the point $(\sqrt{a}, 0, 0)$.

(iii) Show that the solid hyperboloid $x^2 + y^2 - z^2 \leq a$ is a manifold with boundary ($a > 0$).

5. Prove that $\mathbb{R}^n$ and $\mathbb{R}^m$ are not diffeomorphic if $n \neq m$.

6. A *submersion* is a smooth map $f : X \to Y$, between manifolds X and Y, such that df_x is surjective for all $x \in X$. The *canonical submersion* is the standard projection

$$(x_1, \ldots, x_k) \in \mathbb{R}^k \to (x_1, \ldots, x_l) \in \mathbb{R}^l, \qquad \text{for } k \geq l.$$

(i) Let f be a submersion, $y = f(x)$. Show that there exist local coordinates around x and y such that f in these coordinates is the canonical submersion (here $k = \dim X$, $l = \dim Y$).

(ii) Show that submersions are *open maps*, i.e. they carry open sets to open sets.

(iii) If X is compact and Y is connected, show that every submersion is surjective.

(iv) Are there submersions of compact manifolds into Euclidean spaces?

7. Let $f : X \to Y$ be a smooth map and y a regular value of f. Show that the tangent space to $f^{-1}(y)$ at a point x is given by the kernel of $df_x : T_xX \to T_yY$.

8. Prove that the set of all 2×2 matrices of rank 1 is a 3-dimensional submanifold of $\mathbb{R}^4$.

9. For which values of a does the hyperboloid $x^2 + y^2 - z^2 = 1$ intersect the sphere $x^2 + y^2 + z^2 = a$ transversely? What does the intersection look like for different values of a?

10. Let $f : X \to X$ be a smooth map. f is called a *Lefschetz map* if given any fixed point x of f, $df_x : T_xX \to T_xX$ does not have 1 as an eigenvalue. Prove that if X is compact and f is Lefschetz, then f has only finitely many fixed points.

11. Prove the following theorem due to Frobenius: let A be an $n \times n$ matrix all of whose entries are non-negative. Then A has a non-negative real eigenvalue.

12. A manifold is said to be *contractible* if the identity map is homotopic to a constant map. Show that a compact manifold without boundary is not contractible.

13. Let X be a compact manifold and Y a connected manifold with $\dim Y = \dim X$.
(i) Suppose that $f : X \to Y$ has $deg_2(f) \neq 0$. Prove that f is onto.
(ii) If Y is not compact, prove that $\deg_2(f) = 0$ for all maps $f : X \to Y$.

14. Suppose $f : X \to S^k$ is smooth where X is compact and $0 < \dim X < k$. Let $Z \subset S^k$ be a closed submanifold of dimension $k - \dim X$. Show that $I_2(f, Z) = 0$. (Thus degrees are the only interesting intersection numbers on spheres.)
[Hint: Sard's theorem.]

15. (i) Prove that the boundary of a manifold with boundary is a manifold without boundary.
(ii) Show that the square $[0, 1] \times [0, 1]$ is not a manifold with boundary.

16. (i) Let $\lambda : \mathbb{R} \to \mathbb{R}$ be given by $\lambda(x) = e^{-1/x^2}$ for $x > 0$ and $\lambda(x) = 0$ for $x \leq 0$. You know from Analysis I that λ is smooth. Show that $\tau(x) = \lambda(x - a)\lambda(b - x)$ is a smooth function, positive on (a, b) and zero elsewhere ($a < b$).
(ii) Show that

$$\phi(x) := \frac{\int_{-\infty}^{x} \tau}{\int_{-\infty}^{\infty} \tau}$$

is smooth, $\phi(x) = 0$ for $x < a$, $\phi(x) = 1$ for $x > b$ and $0 < \phi(x) < 1$ for $x \in (a, b)$.
(iii) Finally, construct a smooth function from $\mathbb{R}^n$ to the interval $[0, 1]$, that equals 1 on the ball of radius a and zero outside the ball of radius b (here $0 < a < b$).

These functions are very useful for smooth glueings. As an illustration, suppose that $f_0, f_1 : X \to Y$ are smooth homotopic maps. Show that there exists a smooth homotopy $\tilde{F} : X \times [0, 1] \to Y$ such that $\tilde{F}(x, t) = f_0(x)$ for all $t \in [0, 1/4]$ and $\tilde{F}(x, t) = f_1(x)$ for all $t \in [3/4, 1]$. Conclude that smooth homotopy is an equivalence relation.

17. (Morse functions) Let X be a k-dimensional manifold and $f : X \to \mathbb{R}$ a smooth function. A critical point x of f is said to be *non-degenerate* if, in local coordinates around x, the Hessian matrix $\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)$ has non-vanishing determinant. If all the critical points are non-degenerate, f is said to be a *Morse function.*
(i) Show that the condition $\det\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right) \neq 0$ is independent of the choice of chart.
(ii) Suppose now that X is an open subset of $\mathbb{R}^k$. Given $a \in \mathbb{R}^k$, define

$$f_a(x) := f(x) + \langle x, a \rangle,$$

where $\langle x, a \rangle$ denotes the standard inner product in $\mathbb{R}^k$. Show that f_a is a Morse function for a dense set of values of a.
[Hint: consider $\nabla f : X \to \mathbb{R}^k$.]

With a bit more work one can show that the result holds for X *any* smooth manifold. In other words, a 'generic' smooth function is Morse.
(iii) Show that the determinant function on $M(n)$ is Morse if $n = 2$, but not if $n > 2$.